AN

ELEMENTARY TREATISE

ON

OPTICS:

PART II.

CONTAINING

THE HIGHER PROPOSITIONS,

WITH THEIR

APPLICATIONS TO THE

MORE PERFECT FORMS OF INSTRUMENTS

BY

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PREFACE.

The modern history of Optical Science may be considered to commence with the discovery of the law of the refraction of light, by Snellius. To Huygens we are indebted for the discussion of the aberration of pencils refracted by spherical surfaces of media, and he applied his investigations to the construction of his double eye-piece, which remains in high estimation to the present day. By good fortune he secured in it the condition required for achromatism, in addition to the advantages of reduced spherical aberration, to which his views were directed, although the conditions for achromatism in eye-pieces were not discovered until long afterwards.

Sir Isaac Newton having discovered the unequal refrangibility of the differently coloured rays of the spectrum, shewed the great effect of the chromatic dispersion in producing indistinctness of the optical images produced by lenses.

The discovery of the different dispersive powers of different media by Mr. John Dollond, in 1757, and his most successful application of his discovery to the construction of the object-glasses of telescopes, gave a new impetus to the mathematical theory.

We find Clairaut,* D'Alembert,† Euler,‡ and Boscovich,‖ almost simultaneously investigating the properties of refracted

* Mémoires de l'Académie, 1756, 1757, 1762.

† Opuscules, Vol. III. 1764.

‡ The results of many previous papers condensed in his Treatise, Dioptrica, 1769.

‖ Dissertationes quinque ad Dioptricam pertinentes, 1767, and Opuscules, 1785.

pencils, and they have left us their results in laborious and elegant analysis, with, however, great room for improvement in simplicity of method. All these eminent mathematicians investigated the conditions for aplanatism as well as achromatism in the object-glasses of telescopes. Clairaut, in addition, also investigated the aberration in obliquely refracted pencils, and proposed to correct this effect as well as the direct aberration. D'Alembert discusses the primary and secondary foci of a refracted pencil. Boscovich first shewed that the chromatic dispersion of a single eye-glass could not be corrected, except for the point of the image upon the axis, by any form of the achromatic object-glass; and he discovered the construction of the double achromatic eye-pieces, which were hence called Boscovich's eye-pieces. In his later work, he finds the condition of achromatism in combinations of several lenses in eye-pieces. Euler discusses the angle which a given refracted ray makes with the axis of the lens, and compares the relative magnitudes of an object and its image, or successive images, by means of excentrically refracted rays; he also employs the analytical condition for achromatism in combinations of lenses, as used in the eye-pieces of telescopes, but does not keep in view practical cases, during his discussions.

If these various problems had been fully investigated in the neatest and simplest manner of which they admitted, and their results carried out to the cases actually arising in optical instruments, little would have been left for succeeding mathematicians to accomplish. They, however, left the mathematical theory so diffuse and complicated, that few will have had the patience to master entirely their methods, which have hence long remained almost unfruitful.

We do not find that any very important advance in the mathematical theory was made after the above-named illustrious mathematicians, until the Astronomer Royal, Mr. Airy, in his paper, in the "Cambridge Philosophical Transactions," Vol. II. rendered to optical science the great service of applying Euler's analytical method for achromatism to the combinations of lenses for eye-pieces, which had been discussed through circuitous methods by Boscovich. He also undertook, in his paper on the spherical aberrations of eye-pieces,

in the next volume, the discussion of oblique pencils through the primary and secondary focal lines, and the nearest approach to a symmetrical area between them, or the circle of confusion, as the focus of each pencil. He also employed the directions of the axes of excentrical pencils to explain the distortion of images.

These discussions, through the circle of confusion, apply to exceedingly small pencils only when the obliquity is small, because the aberration is neglected, and hence the parts of the field of view out of, but near the axes of lenses, cannot be considered as in the actual case of instruments to have been discussed, although for considerable obliquities, when the aperture is not very large, the results are good approximations. These papers of Mr. Airy induced Mr. Coddington to re-write his Optics, so that his work, published in 1829, is not the third edition of his first Treatise, but a new work, and is the one referred to in the present Treatise.

To find the form of the image in any actual case arising in the use of instruments, it is clear that the oblique aberration must be considered, and the formulæ put in such a workable shape, that the form and curvature of the image can be traced. Amidst the complication which the higher optical formulæ assume, the author has succeeded in obtaining working formulæ, which enable him now for the first time, as he believes, to trace the forms of the images, and to find the lenses possessing the desirable properties of more correct images for various given cases, and hence to discuss the properties of eye-pieces more accurately.

The effects of the oblique aberration in the achromatic lenses constituting the powers of the achromatic microscope, were discovered from experiment by Mr. Lister, and published in his paper, in the "Philosophical Transactions" for 1829; and the astonishing advance towards perfection which the microscope has in consequence attained, unaided by mathematical theory, has been a reflection on the state of mathematical physics. It was clear that no creditable Treatise on Optics could now be undertaken without discussing Mr. Lister's discoveries, and if possible to find new properties which experiment could not be expected to reach without

the mathematical theory. To some extent the author considers he has filled up this desideratum, and carried theory again in advance of practice.

The old approximations for the aberration applied sufficiently accurately to the object-glasses of telescopes, where the refracted pencil seldom exceeds 3^0 or 4^0 in angular diameter; but as the refracting microscopes are now constructed so that their object-glasses bring very accurately, incident pencils of upwards of 90^0 angular diameter, to a focus, the second approximations can apply only imperfectly to such cases, and hence the author considered that the formulæ for the third approximations ought now to be investigated.

From their increased complexity, these formulæ may not be immediately applied to practice, yet in some future improved state of the world of science, when the mathematician, the computer, and the working optician, shall be brought into harmonious action together, from the necessity to secure further advance, the author trusts they will be found useful.

It has been said in optical treatises, that although a parabolic mirror would give a correct image of a star at the focus, yet the image would not be correct when out of the axis; this seems to leave room to doubt the great advantage of a parabolic over a spherical mirror in the Newtonian telescope, and hence it became desirable to discuss the aberrations of ellipsoidal and parabolic mirrors for oblique pencils, as will be found performed in this work.

The author found that the expressions for the reflexion and refraction of pencils at spherical surfaces could be carried one step further without approximation, than has hitherto been done, and he has hence availed himself of that method. He has studied each proposition, to find what he considered the simplest method of treatment, knowing that to many students, optical formulæ seem sufficiently discouraging without any unnecessary complications or developments. For this reason also he has omitted propositions which had less immediate practical bearing.

The notation introduced by Mr. Coddington must now be con-

sidered so far established in this country, that only partial deviations from it should be used. To his last Treatise, which will long be held as the foundation on which succeeding treatises have been constructed, the student is referred who wishes for more extensive study.

To Sir John Herschel also the scientific world is greatly indebted for his excellent Treatise on 'Light,' in the "Encyclopædia Metropolitana." The student will there find many optical developments not to be found in elementary treatises.

LONDON,
DECEMBER, 1850.

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PART II.

CHAPTER I.

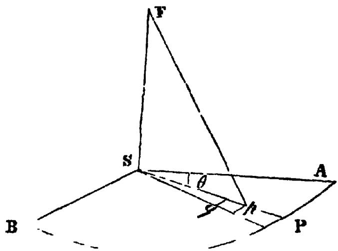
ON APPLICATIONS OF PHOTOMETRY.

REFERRING to the introductory chapter, PART I.; we have the intensity of the light, falling on any surface, measured by the quantity of light, or the number of equally bright rays, which fall on a unit of area, supposing the intensity uniform: and the intensity at different distances from the same luminous point varies inversely as the square of the distance.

Let I represent the intensity at a distance unity from a luminous origin, and consequently $4\pi I$ represents the whole radiant light, since $4\pi r^2$ is the area of the surface of a sphere, and here $r=1$, then the intensity at a distance D is $\frac{I}{D^2}$, and the quantity of light falling directly on a small plane area α at the distance D, is $\alpha \cdot \frac{I}{D^2}$. If the light falls obliquely on the small plane area, and at an angle of incidence i, then the quantity falling upon it is, $\frac{I}{D^2} \cdot \alpha \cdot \cos. i$. (*Cor.* Art. 3, PART I.)

If now we take α an elementary area upon any illuminated surface, and find the expression for the quantity of light falling upon it, the integral between the proper limits, will give us the quantity of light falling on the whole surface.

ARTICLE 1. PROPOSITION. *To find the quantity of light from a luminous point, directly above the pole, which falls on a plane sectorial area.*



Let ASB be the sectorial area; F the position of the luminous origin in the perpendicular to its surface SF; and the height $SF=h$. Let pq be an elementary area, of which the polar co-ordinates are $Sp=r$, $ASP=\theta$, and the area of $pq=r\,.\,d\theta\,.\,dr$. Then the distance $Fp=\sqrt{h^2+r^2}$, and the angle of incidence $=i=\angle pFS$. Using the notation just explained, we have, the quantity of light falling on $pq=\frac{I}{D^2}\,.\,\alpha\,.\cos.\,i$

$$=\frac{I}{h^2+r^2}\,rd\theta\,.\,dr\,\frac{h}{\sqrt{h^2+r^2}}$$

$$=Ih\frac{rdr\,.\,d\theta}{(h^2+r^2)^{\frac{3}{2}}},$$

and the whole light falling on the sectorial area $ASB=$

$$Ih\iint\frac{rdr\,.\,d\theta}{(h^2+r^2)^{\frac{3}{2}}}$$

which being integrated between the given limits, when the polar equation of the curve APB is known, gives the quantity of light required.

EXAMPLE 1. Let the sectorial area be a quadrant of a circle of which the radius is a, then the quantity of light falling upon it is given by the expression

$$\tfrac{1}{2}\pi I\left\{1-\frac{h}{\sqrt{h^2+a^2}}\right\}$$

Ex. 2. If F is the place of a light at a distance h above the center of a circular table, the light falling on the table

$$=2\pi I\left\{1-\frac{h}{\sqrt{h^2+a^2}}\right\}$$

$=2\pi I\times$ versed sine of half the angle subtended by the table at F.

Ex. 3. If a luminous point be in the middle of the axis of a right cylinder, we find the light falling on each end by the last example, and subtracting from the whole radiant light, we have the quantity falling on the concave surface

$=4\pi I\times$ cosine of half the angle subtended by each end.

Ex. 4. The light falling on any solid of revolution from a luminous point (F) in its axis is found, as in Ex. 2.

$=2\pi I\times$ versine of half the angle of the cone with vertex F, which envelopes the solid.

Art. 2. Prop. *To find the intensity of the transmitted light at a given depth in a dense medium, when the proportion lost by dispersion and absorption in passing through a unit of depth, is known.*

Let I = the intensity of the light at entering the medium.
„ I_1 = ... at a depth 1.
„ I_2 = ... 2.
„ I_3 = ... 3.
„ I_t = ... t.

Let $\frac{1}{m}$ be the proportion lost in traversing a unit of depth;

and $M=1-\frac{1}{m}$,

then
$$I_1=I\left(1-\frac{1}{m}\right)$$
$$I_2=I_1\left(1-\frac{1}{m}\right)=I\left(1-\frac{1}{m}\right)^2$$

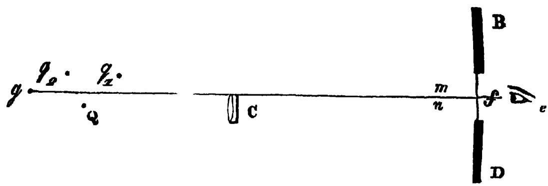
$$I_3=I_2\left(1-\frac{1}{m}\right)=I\left(1-\frac{1}{m}\right)^3$$

and similarly for any depth t,

$$I_t=I\left(1-\frac{1}{m}\right)^t$$
$$=I\,.\,M^t.$$

By this formula we can find the portion of the sun's light which penetrates the water of the ocean to a given depth, when the part lost in passing through any given space has been measured. It also suffices to determine the part transmitted by lenses of given thicknesses, when the effect of the glass of which they are made is known; the reflexion at the surfaces being separately taken into account.

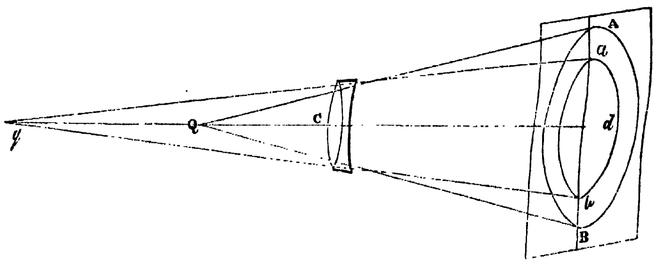
ART. 3. PROP. *To investigate a method of measuring the light transmitted by a given object-glass of a telescope.*



Let the figure represent a horizontal section of a photometer through the small flames of the lamps Q and q_1, which are used; gf representing the horizontal section of a parallelogram of pasteboard 40 to 50 inches long and 2 or 3 inches broad, which is blackened on both sides, or still better, covered with the most non-reflecting substance, black cotton velvet. An upright screen, whose horizontal section is BD, at right angles to gf, has an aperture at mn covered with the thin tissue paper which is used to take copies of written letters by pressure. This aperture being bisected by the upright end of the pasteboard at f, has its respective halves illuminated by the lights Q on one side, and q_1 or q_2 on the other side of the pasteboard, and the equality of the illuminations is judged by an eye at e, there being no other lights present but those of the photometer. Let Q be the determined position of one of the flames, and q_1 that of the one on the other side of the

pasteboard when they illuminate their respective halves of the tissue paper equally. Let then the object-glass C be interposed between Q and fn; the eye will now see the illuminations on the tissue paper to be unequal, and the other flame will require to be moved to some new position as q_2 to make them again equal. The position of q_2 will depend on the condensing power of the lens, and on the portion of light lost in passing through it; the former effect being calculated from the focal length and the position of Q, the latter is determined from the results of the experiment.

To find the intensity which the transmitted light would have had if it had not been condensed into a smaller area by the effect of the lens, let Q in the annexed figure represent the position of



the flame, C that of the lens, and AdB the place of the screen; let $QC=u$, $Cd=d$, $f=$ the focal length of the lens, and $a=$ radius of its aperture. The light diverging from Q which falls on the lens, would have illuminated a circular area AB if the lens had not been interposed, but when emergent from the lens it will diverge from the focus q, which is conjugate to Q, and illuminate the circular area ab. If $qC=v$ we have (PART I. Art. 65.)

$$\frac{1}{v}=\frac{1}{u}-\frac{1}{f}$$

and $$\frac{\text{area of circle } ab}{\text{area of circle } AB}=\frac{(ad)^2}{(Ad)^2}$$

$$=\frac{\left(a\cdot\frac{qd}{qC}\right)^2}{\left(a\cdot\frac{Qd}{QC}\right)^2}$$

$$=\left(\frac{1+\frac{d}{v}}{1+\frac{d}{u}}\right)^2$$

$$=\left(1-\frac{d}{f\left(1+\frac{d}{u}\right)}\right)^2.$$

Now, $\frac{\text{intensity of the light if spread over } AB}{\text{intensity of the same light over } ab}=\frac{\text{area of circle } ab}{\text{area of circle } AB}$

$$=\left(1-\frac{d}{f\left(1+\frac{d}{u}\right)}\right)^2$$

and, referring to the first figure, we have from the experimental result

$$\frac{\text{intensity of the light condensed on } ab}{\text{intensity of the light direct from } Q}=\frac{(q_1m)^2}{(q_2m)^2}$$

Compounding with the previous result we have

$$\frac{\text{intensity of the transmitted light if not condensed}}{\text{intensity of the direct light}}=\left(\frac{q_1m}{q_2m}\right)^2\left(1-\frac{d}{f\left(1+\frac{d}{u}\right)}\right)^2$$

By this method a very fine double achromatic object-glass of four inches aperture and six feet focal length was found to transmit about sixty-six rays out of every one hundred incident. As the apertures of object-glasses of telescopes are increased, the thickness of the lenses must be increased also, and hence the loss of light will be greater than in smaller ones. If the crown or plate glass of which the convex lens is made has much colour, the loss of light will be much increased.

ART. 4. PROBLEM. *To compare the intensity of the light of the heavenly bodies beyond the limits of the atmosphere, with that at the earth's surface.*

To solve this problem rigorously we should have given the density of the atmosphere at all altitudes to which it extends, and the portion of light transmitted (differing with different seasons and

climates) through given spaces for any given density, but we may expect to obtain a good approximation by taking the height to which the atmosphere would extend, if its density, at all altitudes, were the same as at the earth's surface, at the mean; and supposing that light is transmitted through this homogeneous atmosphere as it is at the earth's surface.

Bouguer concluded from his experiments, that light was diminished in the ratio 2500 to 1681 in passing through 7469 toises (9.046 English miles) of dense air; which was nearly the ratio of the sun's light traversing the atmosphere at the summer and winter solstices at Croisic, his place of residence.

To apply this to the formula $I_t = IM^t$ of Art 2, with t expressed in miles, we have

$$\frac{I_{9.046}}{I} = \frac{1681}{2500} = M^{9.046}$$

which gives

$$M = \left(\frac{1681}{2500}\right)^{\frac{1}{9.046}}$$

$$= .95707.$$

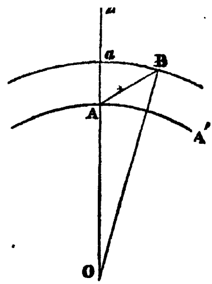
Supposing that the earth's atmosphere, if homogeneous, would extend to the height of 4.9 miles, and I were the intensity of light from a heavenly body beyond that limit, we have for the intensity at the earth's surface, after traversing the atmosphere vertically,

$$I_{4.9} = IM^{4.9}$$
$$= I(.80655)$$

or nearly one-fifth of the light traversing vertically the earth's atmosphere is absorbed, or dispersed.

To find the proportion absorbed when a heavenly body has any given zenith distance, we must find the distance t through which the light passes within the homogeneous atmosphere, and apply it in the preceding formula.

Let O be the center of the earth, AA' the surface, z the zenith of the place A, in the radius OA produced, Aa = the height of the



homogeneous atmosphere determined from the formula

$$Aa = \frac{h\rho}{\rho'}$$

where h = the height of the barometer
ρ = the density of the mercury,
ρ' = . . . of the air,

then AO the radius of the earth being known; $OB = Oa$, and the zenith distance zAB being also known, we have $AB = t$ by solving the triangle OAB.

The formula $\frac{I_t}{I_{t'}} = \frac{M^t}{M^{t'}} = M^{(t-t')}$ gives the ratio of the intensities for two different altitudes when t and t' have been found.

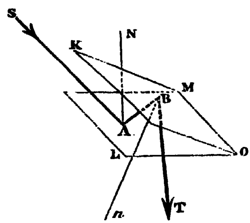
CHAPTER II.

ON THE REFLEXION OF LIGHT BY PLANE AND CURVED MIRRORS.

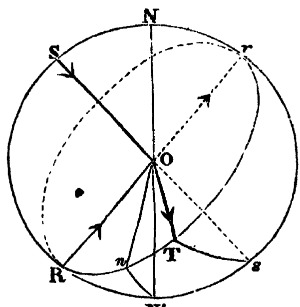
In this chapter we have to discuss the higher propositions of Catoptrics which were omitted, or only mentioned, in Part I.

Art. 5. Prop. *To find the direction of a ray of light after being reflected by two plane mirrors, the planes of reflexion being any whatever.*

Let SA, AB, BT be the course of a ray which is reflected at A by the mirror LM, and at B by the mirror KO; NA, nB being the normals to the mirrors respectively.



If SA were one of a pencil of parallel rays, the reflected rays would be all parallel to AB; and similarly after the second reflexion each ray would be parallel to BT; so that the deviation after the two reflexions would be the same for each ray, and independent of the points of incidence on the two mirrors. Let us suppose that A is indefinitely near to O, and that SO in the lower figure represents the incident ray, ON the normal, and ROr the direction of the reflected ray. Let On parallel to Bn represent the normal to the second mirror, and OT the direction of the ray after being twice reflected.



Let a sphere with radius unity be described round O, and let the lines above-named be radii, the plane of the first reflexion

being in the plane of the paper, and the plane of the second inclined to it. Produce SO to s and NO to N'; and join T and s, n and N' by arcs of great circles. Then the deviation of the twice reflected ray is the angle sOT measured by the arc Ts; and the acute angle between the mirrors equals the angle $N'On$ between the normals, and is measured by the arc nN'.

Let angle of first incidence $= \angle SON = i_1$
. . . second . . . $= \angle ROn = i_2$
. . . between the mirrors $= \angle N'On = \alpha$
. . . between the plane of first incidence and the plane $N'On$ which is perpendicular to both mirrors $= \angle RN'n = \theta$
. . . of deviation $= \angle sOT = D$.

Then in the spherical triangles sRT, $N'Rn$ we have

$$\text{arc } RT = 2i_2,\ Rs = 2i_1,\ Ts = D,\ N'n = \alpha,$$

$$\text{and } \cos. sRT = \frac{\cos. D - \cos. 2i_1 . \cos. 2i_2}{\sin. 2i_1 \sin. 2i_2}$$

$$= \cos. N'Rn = \frac{\cos. \alpha - \cos. i_1 \cos. i_2}{\sin. i_1 \sin. i_2}$$

whence

$$\cos. D = (\cos.\alpha - \cos. i_1 \cos. i_2)\frac{\sin. 2i_1 \sin. 2i_2}{\sin. i_1 \sin. i_2} + \cos. 2i_1 \cos. 2i_2$$

also from spherical triangle $N Rn$ we have

$$\cos. i_2 = \cos. \alpha \cos. i_1 + \sin. \alpha \sin. i_1 \cos. \theta$$

so that D is completely determined when i_1, α, and θ are given.

Expanding the sines and cosines of $2i_1$, and $2i_2$, and substituting the value of $\cos. i_2$ we find after the reductions

$$\cos. D = 4 \cos. \alpha \cos. i_1 \cos. i_2 - 2 \cos.^2 i_2 - 2 \cos.^2 i_1 + 1.$$
$$= \cos. 2\alpha + 2 \sin.^2 \alpha \sin.^2 i_1 \sin.^2 \theta.$$

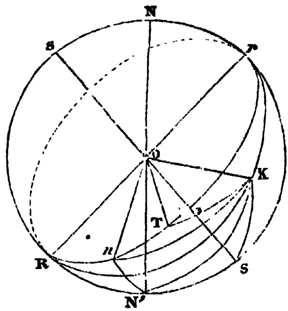
If $\theta = 0$ we have the case of Art. 11. PART I. and

$$D = 2\alpha$$

as there determined.

Art. 6. Prop. *To show that the incident, the once reflected, and twice reflected, rays are equally inclined to the line of intersection of the mirrors.*

Let the letters in the annexed figure represent the same points as in the last; and take K, a point on the sphere, 90^0 distant from n and N'; then the radius OK is the direction of the line of intersection of the mirrors.



Draw the great circle RKr, and join K, T; K, n; K, N'; and K, s; by arcs of great circles.

In the triangles KnR, KnT; since $Kn = 90^0$ we have

$$\cos. KnR = \frac{\cos. KR}{\sin. Rn} = -\frac{\cos. Kr}{\sin. nT}$$

$$= -\cos. KnT = -\frac{\cos. KT}{\sin. nT}$$

$$\therefore\ Kr = KT.$$

In the same way from the triangles $KN'R$, $KN's$ we have

$$\cos. KN'R = \frac{\cos. KR}{\sin. RN'} = -\frac{\cos. Kr}{\sin. N's}$$

$$= -\cos. KNs = -\frac{\cos. Ks}{\sin. N's}$$

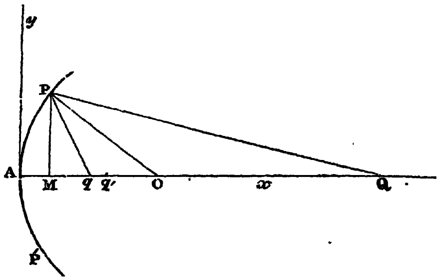
$$\therefore\ Ks = Kr = KT$$

or the directions Os, Or, OT of the incident, the once reflected, and twice reflected rays, make equal angles with OK the line of intersection of the mirrors.

Art. 7. Prop. *To find the aberration of a given ray of a pencil which is incident directly on a* concave *spherical mirror.*

Let PAP' be a section of the mirror, of which O is the center of the curvature, and A the center of the aperture. Let Q be the

focus of the incident rays, QOA the axis of the pencil incident perpendicularly on the mirror at A; which being taken for the origin of co-ordinates; and QP being any ray incident at P of which the co-ordinates are $AM=x$, $PM=y$, let Pq be the reflected ray: then if q_1 be the focus conjugate to Q for a very small direct pencil, as found in Art. 19. PART I., the distance $q_1\,q$ is called the longitudinal aberration of the ray QPq, which is to be found.



Let $QA=u$, $AO=r$, $Aq=v$, $Aq_1=v_1$; the equation of the circle of which PAP' is an arc, is

$$y^2=2rx-x^2=x(2r-x)$$

and since x is always small in the mirrors of reflecting *telescopes*, we may neglect x in respect of $2r$, or neglect x^2 in respect of $2rx$, and use approximately, for them,

$$x=\frac{y^2}{2r} \text{ nearly.}$$

If we solve the above equations in terms of x, we find the series $x=\frac{y^2}{2r}+\frac{y^4}{8r^3}+\frac{y^6}{16r^5}+$ &c.

For the mirrors of reflecting *microscopes* the approximation $x=\frac{y^2}{2r}$ is not always admissible.

Since OP bisects the angle QPq, we have by Euclid, book vi. prop. 3,

$$\frac{qO}{qP}=\frac{QO}{QP} \quad \ldots\ldots \quad (1)$$

and in the triangle QPM, we have

$$\begin{aligned} QP^2 &= QM^2 + PM^2 \\ &= (u-x)^2 + y^2 \\ &= (u-x)^2 + 2rx - x^2 \\ &= u^2\left\{1 - \frac{2x}{u^2}(u-r)\right\} \end{aligned}$$

or
$$QP = u\sqrt{1 - 2x\frac{r}{u}\left(\frac{1}{r} - \frac{1}{u}\right)}$$

In the triangle qPM, we have

$$\begin{aligned} qP^2 &= qM^2 + PM^2 \\ &= (v-x)^2 + y^2 \\ &= v^2 + 2x(r-v) \end{aligned}$$

and
$$\begin{aligned} qP &= v\sqrt{1 + \frac{2x}{v^2}(r-v)} \\ &= v\sqrt{1 + 2x\,.\,\frac{r}{v}\left(\frac{1}{v} - \frac{1}{r}\right)} \end{aligned}$$

Substituting in the equation (1) we have

$$\frac{r-v}{v\sqrt{1+2x\,.\,\frac{r}{v}\left(\frac{1}{v}-\frac{1}{r}\right)}} = \frac{u-r}{u\sqrt{1-2x\,.\,\frac{r}{u}\left(\frac{1}{r}-\frac{1}{u}\right)}}$$

or dividing by r, both sides of the equation,

$$\left(\frac{1}{v} - \frac{1}{r}\right)\left\{1 + 2x\,.\,\frac{r}{v}\left(\frac{1}{v} - \frac{1}{r}\right)\right\}^{-\frac{1}{2}} = \left(\frac{1}{r} - \frac{1}{u}\right)\left\{1 - 2x\,.\,\frac{r}{u}\left(\frac{1}{r} - \frac{1}{u}\right)\right\}^{-\frac{1}{2}} \quad (2)$$

when x is so small that we may neglect the terms into which it enters as a multiplier, we have, as in PART I.,

$$\frac{1}{v_1} - \frac{1}{r} = \frac{1}{r} - \frac{1}{u}, \qquad \text{or} \qquad \frac{1}{v_1} = \frac{2}{r} - \frac{1}{u}$$

substituting these in the term involving x, supposed small but not negligible, we have

$$\frac{1}{v}=\frac{1}{r}+\left(\frac{1}{r}-\frac{1}{u}\right)\left\{1-2x\frac{r}{u}\left(\frac{1}{r}-\frac{1}{u}\right)\right\}^{-\frac{1}{2}}.\left\{1+2xr.\left(\frac{2}{r}-\frac{1}{u}\right)\left(\frac{1}{r}-\frac{1}{u}\right)\right\}^{\frac{1}{2}}$$

$$=\frac{1}{r}+\left(\frac{1}{r}-\frac{1}{u}\right)\left\{1+\frac{xr}{u}\left(\frac{1}{r}-\frac{1}{u}\right)\right\}.\left\{1+xr\left(\frac{2}{r}-\frac{1}{u}\right)\left(\frac{1}{r}-\frac{1}{u}\right)\right\}\text{nearly}$$

by extracting the square roots and omitting terms with x^2, x^3, &c.

$$=\frac{2}{r}-\frac{1}{u}+xr\left\{\left(\frac{1}{r}-\frac{1}{u}\right)^2\left[\frac{1}{u}+\left(\frac{2}{r}-\frac{1}{u}\right)\right]\right\}$$

neglecting the term with x^2

$$=\frac{2}{r}-\frac{1}{u}+2x\left(\frac{1}{r}-\frac{1}{u}\right)^2$$

If we take the approximate value of $x=\dfrac{y^2}{2r}$

we have $$\frac{1}{v}=\frac{2}{r}-\frac{1}{u}+\frac{y^2}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2$$

which gives the value of v to a *second* approximation; and when more correct values still are required, they may be obtained from (2) by successive substitutions for v in the term involving x, and extracting numerically the square root of the factors where it is indicated.

When the second approximation only is wanted, the following is the simplest procedure. By neglecting at once the powers of x above the first at each step, and substituting the first approximate value, in the small term, we have

$$QP=u\left\{1-2x\frac{r}{u}\left(\frac{1}{r}-\frac{1}{u}\right)\right\}^{\frac{1}{2}}$$

$$=u\left\{1-\frac{xr}{u}\left(\frac{1}{r}-\frac{1}{u}\right)\right\}\text{ nearly.}$$

$$qP=v\left\{1+2x\frac{r}{v}\left(\frac{1}{v}-\frac{1}{r}\right)\right\}^{\frac{1}{2}}$$

$$=v\left\{1+\frac{xr}{v}\left(\frac{1}{v}-\frac{1}{r}\right)\right\}\text{ nearly.}$$

and substituting in $\dfrac{qO}{qP}=\dfrac{QO}{QP}$

we have $\dfrac{r-v}{v\left\{1+\dfrac{xr}{v}\left(\dfrac{1}{v}-\dfrac{1}{r}\right)\right\}}=\dfrac{u-r}{u\left\{1-\dfrac{xr}{u}\left(\dfrac{1}{r}-\dfrac{1}{u}\right)\right\}}$

$$\therefore \frac{1}{v}-\frac{1}{r}=\left(\frac{1}{r}-\frac{1}{u}\right)\left\{1+\frac{xr}{v}\left(\frac{1}{v}-\frac{1}{r}\right)\right\}\cdot\left\{1+\frac{xr}{u}\left(\frac{1}{r}-\frac{1}{u}\right)\right\} \text{nearly,}$$

$$=\left(\frac{1}{r}-\frac{1}{u}\right)\left\{1+xr\left[\frac{1}{v}\left(\frac{1}{v}-\frac{1}{r}\right)+\frac{1}{u}\left(\frac{1}{r}-\frac{1}{u}\right)\right]\right\}$$

$$=\frac{1}{r}-\frac{1}{u}+xr\left\{\left(\frac{2}{r}-\frac{1}{u}+\frac{1}{u}\right)\left(\frac{1}{r}-\frac{1}{u}\right)^2\right\}$$

$$\therefore \frac{1}{v}=\frac{2}{r}-\frac{1}{u}+2x\left(\frac{1}{r}-\frac{1}{u}\right)^2$$

$$=\frac{1}{f}-\frac{1}{u}+\frac{y^2}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2.$$

The aberration $q_1q=v_1-v=\delta v$ say,

and $$\frac{1}{v}-\frac{1}{v_1}=\frac{v_1-v}{v_1\,v}=\frac{\delta\,v}{v_1v}$$

therefore δv is known when $\dfrac{1}{v}$ and $\dfrac{1}{v_1}$ have been found; and for a second approximation we have, since $v_1v=v_1{}^2$ nearly;

$$\delta v=v_1v\left(\frac{1}{v}-\frac{1}{v_1}\right)=\frac{y^2v_1{}^2}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2.$$

If the incident pencil had consisted of parallel rays, or if $\dfrac{1}{u}=0$, then the aberration $=\delta v=\dfrac{y^2}{4r}$.

This result may be very easily obtained geometrically, as half the versed-sine of the arc of the mirror nearly.

If we had taken q for the focus of the incident rays, we should have found, by a similar procedure,

$$\frac{1}{u}=\frac{2}{r}-\frac{1}{v}+2x\left(\frac{1}{v}-\frac{1}{r}\right)^2$$

and the aberration δu would still have been measured towards A from the first approximate focus; since we have

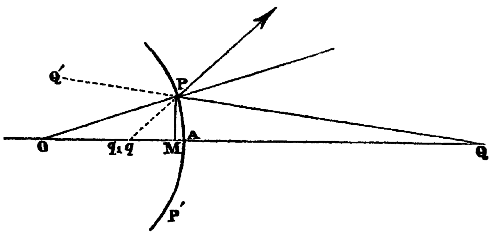
$$\frac{1}{u} > \frac{1}{u_1}$$

and
$$\delta u = \frac{y^2 u_1^2}{r}\left(\frac{1}{v} - \frac{1}{r}\right)^2,$$

which is identical with the previous expression by putting v for u; and we see that the aberration increases as $v_1^2\left(\frac{1}{r} - \frac{1}{u}\right)^2$ or $u_1^2\left(\frac{1}{v} - \frac{1}{r}\right)^2$ increase, and vanishes when Q and q meet at O. It also, for a given position of the focus of the incident rays, varies as (y^2) the square of the distance of the point of incidence from the axis of the mirror, in small pencils.

ART. 8. PROP. *To find the aberration of a given ray of a pencil which is incident directly on a* convex *spherical mirror.*

Let Q be the focus of the pencil incident on the convex mirror PAP'; and let the other letters in the annexed figure indicate the corresponding points to those in the last proposition, respectively.



Then any ray QP being produced to Q', the exterior angle $Q'Pq$ is bisected by the radius OP, and by Euclid, book vi. prop. A.

$$\frac{qO}{qP} = \frac{QO}{QP} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also taking $AM = x$, $PM = y$ we have by proceeding in the same manner as in the last prop.

$$QP = u\sqrt{1+2x\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)}$$

$$qP = v\sqrt{1+2x\frac{r}{v}\left(\frac{1}{v}-\frac{1}{r}\right)}$$

and the equation (1) becomes

$$\frac{r-v}{v\sqrt{1+2x\frac{r}{v}\left(\frac{1}{v}-\frac{1}{r}\right)}} = \frac{u+r}{u\sqrt{1+2x\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)}}$$

or
$$\frac{1}{v}-\frac{1}{r}=\left(\frac{1}{r}+\frac{1}{u}\right)\left(\frac{1+2x\frac{r}{v}\left(\frac{1}{v}-\frac{1}{r}\right)}{1+2x\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)}\right)^{\frac{1}{2}} \quad \ldots\ldots (2)$$

neglecting for the first approximation, the terms involving x, we have, as in Art. 20, PART. I.,

$$\frac{1}{v_1}-\frac{1}{r}=\frac{1}{r}+\frac{1}{u}, \quad \frac{1}{v_1}=\frac{2}{r}+\frac{1}{u}$$

and substituting these values in the coefficient of x and expanding, we find

$$\frac{1}{v}=\frac{2}{r}+\frac{1}{u}+2x\left(\frac{1}{r}+\frac{1}{u}\right)^2 \text{ nearly}$$

$$=\frac{2}{r}+\frac{1}{u}+\frac{y^2}{r}\left(\frac{1}{r}+\frac{1}{u}\right)^2 \quad \ldots\ldots$$

If we take q_1 the conjugate focus to Q, from the first approximation, putting $Aq_1=v_1$ we have

$$\frac{1}{v}>\frac{1}{v_1} \quad \text{and} \quad \therefore v_1>v$$

or the aberration is again towards A from q_1

and
$$\delta v=\frac{y^2v_1^2}{r}\left(\frac{1}{r}+\frac{1}{u}\right)^2.$$

If the incident pencil converged to q, we should have by substituting in (2), the first approximate value of u, in the small term

$$\frac{1}{u}=\frac{1}{v}-\frac{2}{r}-2x\left(\frac{1}{v}-\frac{1}{r}\right)^2$$
$$=\frac{1}{v}-\frac{2}{r}-\frac{y^2}{r}\left(\frac{1}{v}-\frac{1}{r}\right)^2$$

putting u_1 the first approximate value of u, we see that now $\frac{1}{u}<\frac{1}{u_1}$ and $\therefore u>u_1$, or the aberration is now from A, and

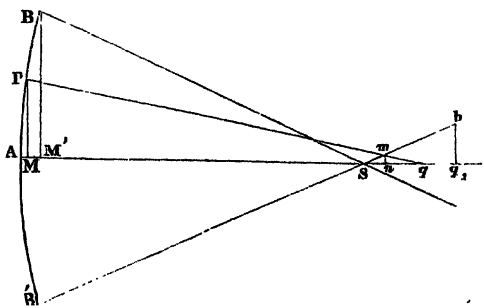
$$\delta u=\frac{y^2u_1^2}{r}\left(\frac{1}{v}-\frac{1}{r}\right)^2$$

It will be seen that the results in this article might have been deduced from those of the previous one by taking the lines with contrary signs when they have to be measured in opposite directions to that of the case taken as standard, as at page 33, Part I., and in the same way the results for other cases may be found.

It will be seen that, in convex as well as concave mirrors, the aberration is measured from the mirror when the focus of the incident rays lies between the mirror and principal focus; and in all other cases is measured towards it. The aberration vanishes when Q and q meet at O.

Art. 9. Prop. *To find* the least circle of aberration *when a pencil of rays is reflected directly at a spherical mirror.*

We suppose the aperture of the mirror to bear such a proportion to the focal length, that we may use the second approximations of the two last articles.



Let $BPAB'$ be the section of the mirror whose axis is Aq_1; Bs, $B's$ the rays reflected

at the edge of the mirror which intersect the axis in s; q_1 the geometrical focus for rays reflected indefinitely near to A; Pq any other ray of the pencil cutting the axis in q, so that q_1q is the aberration of that ray.

Then if we put $y' = BM' =$ half aperture of mirror,
$y = PM$

we have from the results of the two last propositions

$$\frac{q_1q}{q_1s} = \frac{y^2}{y'^2}$$

or since y', the radius (r) of the curvature of the mirror, and u the distance of the luminous origin are given, therefore the aberration q_1s of the extreme ray is given; let $q_1s = a$,

then $$q_1q = a\frac{y^2}{y'^2}$$

let mn be the radius of the least circle through which the rays Bs, $B's$ and Pq pass; we have from the triangle msn, $BM's$,

$$\frac{mn}{sn} = \frac{BM'}{M's}, \text{ or } mn = \frac{BM'}{M's} \cdot sn$$

and from triangles mqn, PqM,

$$\frac{mn}{nq} = \frac{PM}{qM}, \text{ or } mn = \frac{PM}{qM} \cdot nq$$

Now since the aberration is by supposition very small, so that we use the second approximations, we have $M's = Mq = Aq_1 = v_1$ very nearly; therefore equating the values of mn, we have

$$\frac{y'}{v_1} \cdot sn = \frac{y}{v_1} \cdot nq$$

and $$nq = sq_1 - q_1q - sn$$
$$= a - a\frac{y^2}{y'^2} - sn$$

substituting this value of nq, we have

$$sn = \frac{ay}{y'^2}(y' - y)$$

and $$mn = \frac{y'}{v_1} sn$$
$$= \frac{ay}{v_1 y'} (y' - y)$$

Now when mn is a maximum in respect of the varying position of P, we have by *the differential calculus*,

$$\frac{d(mn)}{dy} = 0$$
$$= \frac{a}{v_1 y} (y' - 2y)$$
$$\therefore \; y' - 2y = 0, \quad \text{or } y = \frac{y'}{2}$$

substituting this value, we have

$$sn = \frac{ay}{y'^2} (y' - y)$$
$$= \frac{a}{4}$$
$$= \frac{q_1 s}{4} \qquad \text{and} \qquad q_1 n = \frac{3}{4} \cdot q_1 s$$

which gives the position of the least circle of aberration;

and its radius $$mn = \frac{y'}{v_1} \cdot sn$$
$$= \frac{y'}{4v_1} q_1 s$$

or substituting for $q_1 s$ its value from Art. 7, we have for a concave mirror the radius of the least circle of aberration

$$= \frac{y'}{4v_1} \cdot \frac{y'^2 v_1^2}{r} \left(\frac{1}{r} - \frac{1}{u}\right)^2$$
$$= \frac{y'^3 v_1}{4r} \left(\frac{1}{r} - \frac{1}{u}\right)^2$$

When the incident pencil consists of parallel rays, or $\frac{1}{u} = 0$, and $v_1 = f = \frac{r}{2}$

the radius of the least circle of aberration $=\frac{y'^3}{8r^2}$

$$=\frac{y'^3}{32f^2}$$

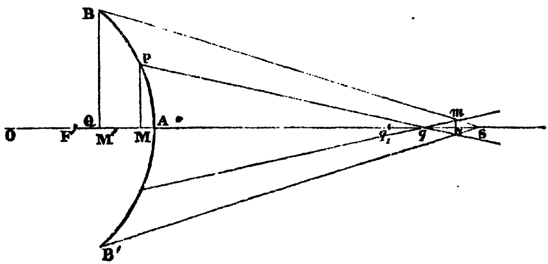
or the diameter $=\frac{y'^3}{4r^2}$

$$=\frac{y'^3}{16f^2}$$

If we draw q_1b perpendicular to the axis and meeting $B's$ in b, then q_1b is called the lateral aberration of the extreme ray, and

$$q_1b=4\,.\,mn$$
$$=\frac{y'^3}{8f^2}$$

Cor. When a converging pencil is incident upon a convex spherical mirror so as to give a *real* focus of the reflected rays, the aberration (see the latter part of Art. 8) is from the vertex of the mirror as in the figure; but by pursuing the same method as above, we find the same forms for the values of sn and mn



or $$sn=\frac{1}{4}q_1s$$

$$mn=\frac{y'}{4v_1}q_1s=\frac{y'^3\,v_1}{4r}\left(\frac{1}{u}-\frac{1}{r}\right)^2$$

By substituting the proper value of q_1s, the result may be adapted to the other cases of reflexion at spherical mirrors.

We are now able to calculate the effect of using spherical mirrors in catoptrical instruments in place of mirrors of the correct figures for no aberration.

EXAMPLE. To find the effect on the distinctness of a seven-feet Newtonian telescope of the proportions used by Sir W. Herschel, when a spherical mirror is used in place of a paraboloidal one.

In this telescope, with which Sir W. Herschel made so many discoveries, the aperture of the large mirror was $6\frac{3}{10}$ inches, and its focal length about 86 inches.

Here we have $y' = 3\frac{3}{20}$ inches, and $r = 172$ inches.

$\therefore$ the longitudinal aberration of the extreme rays $= \dfrac{y'^2}{4r} = \frac{1}{70}$ of an inch nearly.

The diameter of the least circle of aberration $= \dfrac{y'^3}{4r^2} = \frac{1}{3780}$ of an inch, which is less than one-tenth part of the breadth of a hair of the head, taking it as $\frac{1}{300}$ of an inch. If we divide this quantity by the focal length, 86 inches, we find that the diameter of the least circle of aberration subtends at the center of the mirror an angle of $\frac{5}{8}$ of *one second* of a degree; and this would not prevent the telescope separating the images of many difficult double stars,* which are considered most effectual test-objects for telescopes.

If such a telescope will not shew these objects when charged with sufficient magnifying power, we must conclude that the workmanship is defective either in the large or small plane mirror; and that it is useless to speak of a parabolic figure until the mirrors will succeed with them in a satisfactory manner. It is, moreover, to be remembered, that the physical effect of diffraction would remain, even if the mirrors were each of a perfect surface and polish.

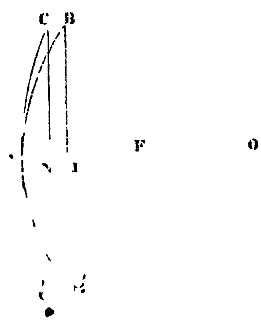
The great difficulty of making the small mirror of the Newtonian telescope very nearly plane, renders it advisable for amateur

* See Herschel's "Astronomy," chapter xii.

telescope makers to construct the Herschelian form first; remembering that by dispensing with the small mirror, one-third of the light is saved, and hence there will be equal brightness with a speculum of considerably less aperture, which, however, should not exceed one-fifteenth to one-twentieth of the focal length.

Art. 10. Prop. *To investigate an expression for the difference of the extreme abscissæ for a spherical mirror, and one formed by the revolution of a conic section about its major axis, when they are of the same aperture and focal length.*

Since the radius of curvature of all the conic sections at the extremity of the major axis equals half the latus rectum, and we must have the spherical mirror the nearest possible to the given ellipsoidal, paraboloidal or hyperboloidal one, therefore the radius of the spherical mirror must equal half the latus rectum of the conic section by which the surface of the other is generated.



Let CAC' be a conic section, of which the focus is F, BAB' an arc of a circle whose center is O and radius is $AO=r$, which is equal to half the latus rectum of CAC'.

Then the general equation of a conic section being

$$y^2 = mx + nx^2$$

where m = latus rectum, and $n=0$ in the parabola, is negative in the ellipse, and positive in the hyperbola; also if a and b be the major and minor axes of the two latter, we have

$$m = 2\frac{b^2}{a}, \quad n = \mp\frac{b^2}{a^2}.$$

If $y^2 = 2rx - x^2$ be the equation of the circular arc, by the above conditions, we have $2r = m$.

Now let the extreme ordinates be equal, or $BM = y = CN$, and let $AM = x$, $AN = x'$; to find the difference $MN = x - x'$, we have

$$y^2 = mx' + nx'^2 = 2rx' + nx'^2$$
$$y^2 = 2rx - x^2$$

subtracting $\quad 0 = 2r(x - x') - x^2 - nx'^2$

or $\quad NM =$ the difference of the abscissæ required

$$= x - x' = \frac{x^2 + nx'^2}{2r}.$$

Since x and x' are very small compared with $2r$

and $\quad y^2 = x(2r - x) = x'(2r + nx')$

we may use the approximate expressions $x = \frac{y^2}{2r}$, $x' = \frac{y^2}{2r}$

and $$NM = \frac{y^4}{8r^3}(1 + n)$$
$$\propto y^4$$

This formula gives a very correct value for NM, until the magnitude of y is much greater than is ever used in telescopes: to apply it to a parabolic mirror in the example of the last article, we have

$$NM = \frac{(3\tfrac{3}{20})^4}{8 \times 172^3} \text{ inches}$$
$$= .0000024 \text{ inch,}$$

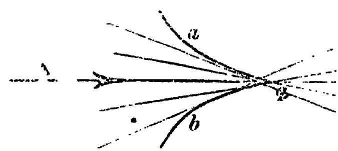
or less than one four hundred thousandth part of an inch. We may readily grant that a spherical mirror should be very accurately ground and polished before a parabolic figure is thought of; and then the quantity to be polished away at each point is proportional to the fourth power of the distance from the vertex.

When the axis of the incident pencil falls obliquely on the mirror, the effects are not symmetrical, but may be classified under two heads: the mirror not being very small nor the obliquity very great, we may term the effect *oblique aberration;* and when the mirror is very small, or the obliquity very great, we may call the effect *confusion.*

We shall be better able to trace the changes which arise as the obliquity is increased, from the least circle of aberration, when the obliquity is nothing, to the formation of the least circle of con-

fusion, when it is very great, by first considering the origin of the caustic lines and caustic surface.

The caustic line being the locus of the intersections of consecutive rays, takes two forms, according as we consider the consecutive rays to have the same plane of incidence, in which case the caustic lies in the same plane; or, as we consider the plane passing through two consecutive rays to be perpendicular to the plane of incidence. We are at present considering only the reflexion by spherical mirrors, and referring to the figures of Art. 9, we see that the caustic for the rays, incident in the plane of the paper, will have two like branches aq_1, bq_1, one above and one below the axis Aq_1, meeting in a cusp at the geometrical focus q_1, as in the annexed figure; and the revolution of either branch round the axis Aq_1 will form the caustic surface. If the *mirror* be cut by a plane perpendicular to the axis, the intersection will be a circle with its center in the axis, and rays incident upon the mirror in this circle will be all similarly related to the axis, and the reflected rays will all intersect it in the same point, which is the vertex of the conical reflected pencil. If we consider the face of the mirror to be made up of such circles, we shall have the reflected light forming a series of cones with their common axis that of the mirror, and the vertices of the cones forming another caustic or *focal line*.



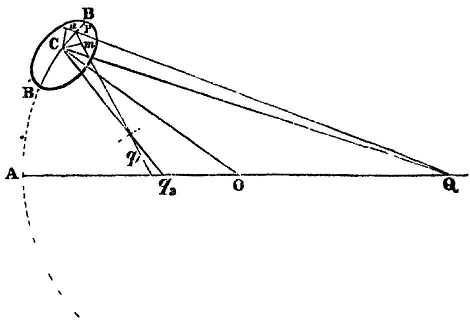
In the cases of accurate reflexion, the caustics are reduced to a point, which is the accurate focus.

Whether we consider the caustic surface to be formed by the revolution of the caustic curve, or by the ultimate intersections of the series of cones which have their bases' circles on the face of the mirror, and their vertices in its axis; we see that the sections of the reflected pencils will not be symmetrical unless the mirror be a segment of a sphere, with its axis coinciding with the axis of the incident pencil.

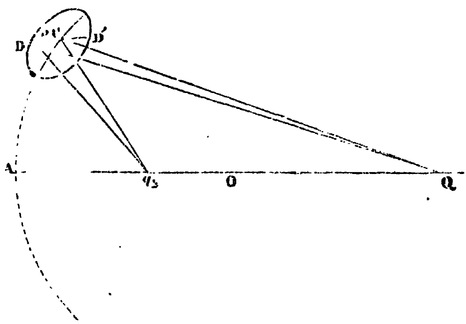
If we could suppose a large mirror reflecting a large direct pencil, and forming the corresponding caustic surface, and then could trace the modifications of the latter, when portions of the *mirror were covered over so as to leave only a small reflecting* surface, we might find the effects in the reflected pencils for different cases, but they will be found in the following propositions better treated by independent methods.

ART. 11. PROP. *To find the form of a pencil reflected* obliquely *by a* small *spherical mirror.*

Let BCB' be a section of the small mirror seen in perspective, which may be considered a portion of a larger one, of which the center is O. Let Q be the focus of the incident rays and draw the line QOA which will be the axis of the supposed larger mirror. Let C be the center of the aperture of the small mirror, QC the axis of the incident oblique pencil, Cq_1q_2 that of the reflected pencil. Let QP be another



incident ray, indefinitely near to QC, reflected in Pq_1; then q_1 being the intersection of the reflected rays, is called the *primary focus* of the pencil. If we take other planes of incidence, which all pass through



the line QOA, above or below the plane of the paper, the reflected rays will have their *primary foci* in a circular arc above or below q_1 respectively; and for the whole of the mirror we shall have formed this arc as the *primary focal line* at q_1.

In the lower figure, let DCD' be a circular arc, which is the intersection of the mirror by a plane perpendicular to QOA; and QC, QP being rays incident on DCD', they will be similarly situated with respect to QOA; then the reflected rays will also be similarly situated with respect to QOA and will intersect it in the same point q_2; or q_2 is the *secondary focus*. If we take other circular arcs in the same manner, higher and lower than DCD', they will have the *secondary foci*, further from or nearer to O than q_2 respectively, and for the whole mirror will form the *line of secondary foci* in AOQ.

To find the distances Cq_1, Cq_2 from the center of the mirror; let $QC=u$, $CO=r$, $Cq_1=v_1$, $Cq_2=v_2$, and the angle of incidence of the axis of the pencil, $QCO=i=$ angle of reflexion q_1CO.

In the upper figure draw the perpendiculars Cn, Cm, upon QP produced, and q_1P respectively: then the indefinitely small triangles CPn, CPm, are equal in every respect, for they have the side PC common, the angles at m and n right angles, and the angles CPn, CPm equals, because they are the complements of the angles of incidence and reflexion at P.

$$\therefore Pn=Pm$$

and
$$\begin{aligned} Pn &= QC-QP = \text{decrement of } u \\ &= -du \\ Pm &= q_1P-q_1C = \text{increment of } v_1 \\ &= dv_1 \\ \therefore dv_1 &= -du. \end{aligned}$$

Join Oq_1; in the triangles QCO, q_1CO, we have

$$\begin{aligned} QO^2 &= QC^2+OC^2-2\,.\,QC\,.\,OC\cos. i \\ &= u^2+r^2-2ru\cos. i \\ q_1O^2 &= q_1C^2+OC^2-2\,.\,q_1C\,.\,OC\cos. i \\ &= v_1^2+r^2-2rv_1\cos. i \end{aligned}$$

here u, v_1 and cos. i vary from C to P, whilst QO and q_1O are constant. $\therefore$ differentiating we have

$$0 = u\,du - r\cos. i\,du - r\,u\,d\,.\cos. i$$
$$0 = v_1\,dv_1 - r\cos. i\,dv_1 - r\,v_1\,d\,.\cos. i$$

equating the values of $d\,.\cos. i$, and omitting the common factor du after substituting $-du$ for dv_1, we have

$$\frac{u - r\cos. i}{r\,u} = \frac{r\cos. i - v_1}{r\,v_1}$$

whence
$$\frac{1}{v_1} = \frac{2\sec. i}{r} - \frac{1}{u}$$

which gives v_1.

Again in the triangle QCq_2, the radius OC bisects the angle QCq_2, and we have

$$\frac{QO}{QC} = \frac{q_2\,O}{q_2\,C}$$

and
$$\therefore\ QO^2 \times q_2C^2 = q_2O^2 \times QC^2$$

or
$$(u^2 + r^2 - 2ru\cos. i)\,v_2^{\,2} = (v_2^{\,2} + r^2 - 2rv_2\cos. i)\,u^2$$

and reducing

$$r\,(v_2^{\,2} - u^2) = 2\,uv_2\cos. i\,(v_2 - u)$$

or $\quad r\,(v_2 + u) = 2\,uv_2\cos. i$, and dividing each term by urv_2

$$\therefore\quad \frac{1}{v_2} = \frac{2\cos. i}{r} - \frac{1}{u}$$

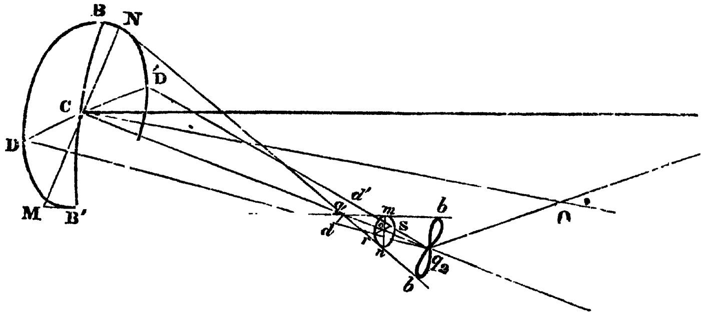
which gives v_2.

Art. 12. Prop. *To find the* least circle of confusion *in a pencil of rays, after oblique reflexion by a spherical mirror.*

As the most interesting case in the oblique reflexion by spherical mirrors, we will first take the circumstances in which it would be applied in the Herschelian telescope. The axis of the eye-tube coinciding with that of the reflected pencil, we must seek the section of the pencil perpendicular to this line, between q_1 and

q_2, which is the nearest to a circle, or when the length equals the breadth.

In the figure, let C be the center of the face of the mirror seen



in perspective; Cq_1q_2 the axis of the reflected pencil; dq_1d' the *primary focal line*; bq_2b' the *secondary focal line*, or section perpendicular to Cq_1q_2 through q_2 which is a narrow figure of eight, since the secondary foci are in the line Oq_2; and let $mrns$ be the least circle of confusion, where mn in the plane Bq_1B' equals rs in the plane Dq_2D'.

Let $BB'=DD'$ the aperture of the mirror $=2a$; the breadth MN of the pencil perpendicular to $Cq_1=2a$ cos. i, because angle $BCN=$ angle $q_1CO=i$, when BB' is small; let $Cq_1=v_1$, $Cq_2=v_2$ as before; and let o in the line Cq_1q_2 be the intersection of mn and rs. We have by similar triangles

$$\frac{dd'}{q_1q_2}=\frac{DD'}{Cq_2}, \text{ or the primary focal line, } dd'=\frac{2a}{v_2}(v_2-v_1)$$

$$\frac{bb'}{q_1q_2}=\frac{MN}{Cq_1}, \text{ or the secondary focal line, } bb'=\frac{2a\cos. i}{v_1}(v_2-v_1)$$

and $$\frac{rs}{q_2o}=\frac{dd'}{q_1q_2}, \text{ or } rs=q_2o\cdot\frac{2a}{v_2}$$

$$\frac{mn}{q_1o}=\frac{bb'}{q_1q_2}, \text{ or } mn=q_1o\cdot\frac{2a\cos. i}{v_1}$$

$$=(v_2-v_1-q_2o)\frac{2a\cos. i}{v_1}$$

but $mn=rs$

$$\therefore \qquad q_2 o \cdot \frac{2a}{v_2} = (v_2 - v_1 - q_2 o)\,\frac{2a \cos. i}{v_1}$$

whence
$$q_2 o = \frac{v_2 \cos. i\,(v_2 - v_1)}{v_1 + v_2 \cos. i}$$

which gives the position of the point o, and when i is small or $\cos. i = 1$ nearly, $q_2 o = \frac{1}{2}(v_2 - v_1)$ nearly; or the least circle of confusion is nearly equally distant from q_1 and q_2.

Again, the diameter of the least circle of confusion $= rs$

$$= q_2 o \cdot \frac{2a}{v_2}$$
$$= \frac{2a \cos. i\,(v_2 - v_1)}{v_1 + v_2 \cos. i}$$

Since the telescope is used to view distant objects, putting $\frac{1}{u} = 0$

we have $\qquad v_1 = \frac{r \cos. i}{2}, \qquad v_2 = \frac{r \sec. i}{2}$

substituting these values

the diameter of the least circle of confusion $= \frac{2a \cos. i\,(\sec. i - \cos. i)}{(\cos. i + 1)}$

$$= 2a\,(1 - \cos. i)$$
$$= 2a \text{ vers. } i$$

If we suppose the aperture to bear the proportion to the focal length, which is found in achromatic telescopes, or $2a = \frac{f}{15} = \frac{r}{30}$, we have $i = 57'$ nearly,
and the least circle of confusion $= 2a$ vers. $57'$

$$= 2a \times .0001375$$

In this proportion a mirror, as in Art. 9, of 86 inches focal length, would have an aperture of $5\frac{11}{15}$ inches, and the least circle of confusion $\qquad = .000788$ inch

$$= \tfrac{1}{1270} \text{ inch nearly,}$$

which is much more than the aberration found in Art. 9, though

still a small quantity; and the aperture might be reduced below this proportion, and still have equal light with the Newtonian form, besides saving the errors of workmanship of the small plane mirror.

Cor. When the incident pencil is conical, with its perpendicular section a circle, and falls obliquely on a larger mirror, so that DD' in the figure $=\lambda$ $=$breadth of the pencil; then $BB'=\lambda$ sec. i. Proceeding as before, we find

$$\text{the primary focal line} \quad =\frac{DD'}{Cq_2} \cdot q_1q_2$$

$$=\frac{\lambda}{v_2}(v_2-v_1)$$

the secondary focal line being a section of the pencil by a plane through q_2 parallel to the tangent plane to mirror at $C=\frac{BB'}{Cq_1} \cdot q_1q_2$

$$=\frac{\lambda \sec. i}{v_1}(v_2-v_1)$$

which give, by the same method as before

$$q_2o=\frac{v_2(v_2-v_1)}{v_1 \cos. i+v_2}$$

and the diameter of the least circle of confusion

$$=\frac{\lambda(v_2-v_1)}{v_1 \cos. i+v_2}.$$

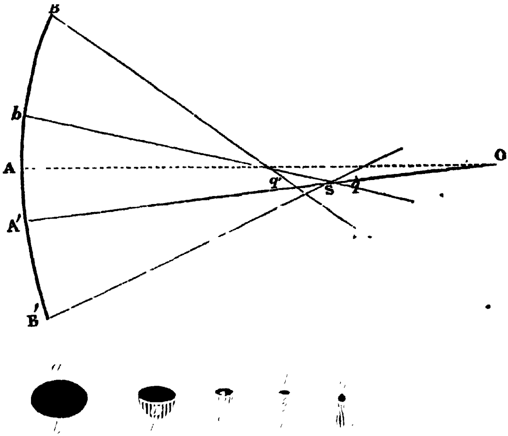
When the incident pencil consists of parallel rays, we find by the same substitutions as before,

$$\text{the diameter of the least circle of confusion} =\frac{\lambda \sin.^2 i}{1+\cos.^3 i}.$$

We have seen in the preceding articles the effect of the figure in a spherical mirror in producing aberration directly, and the effect of a *small* spherical mirror in producing confusion, but an effect of equal or indeed greater consequence has hitherto not been discussed by mathematicians, namely *oblique aberration.*

If OA in the figure is the axis of BAB', a spherical mirror, OA' the

axis of a pencil falling obliquely upon it, and we take $A'b = A'B'$ the *distance of the nearest edge of* the mirror to A', then we may consider a complete circular area with diameter $bA'B'$ to be taken on the face of the mirror, which will have its geometrical focus for small pencils at q, and the rays at its edge cutting the axis



in s: the rays reflected outside this area will cut the axis nearer to A', and we suppose the one reflected from the edge at B to intersect it at q'.

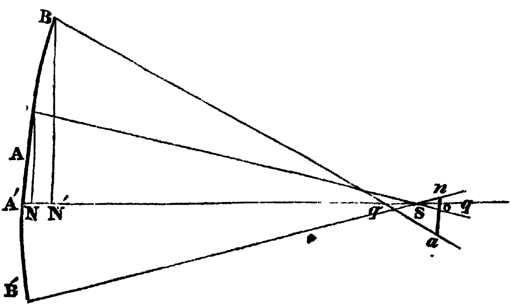
If we now consider the perpendicular sections of the reflected pencil, we shall see that nearer to A' than q' we shall have an oval, like the first of the lower figures in white, most flattened and brightest at a, the upper edge; about q' we shall have a section like the next figure (the nearest approach to the primary focal line which occurs at greater obliquities), the upper part a, in the first figure, has descended to a in the second, by an overlapping of the upper rays, producing a bright upper edge; at the intersection of the lines $B's$ and Bq' produced, we have the parts a and b coinciding, and a little nearer s we have a section, like the middle figure, the part a being the lowest and the part b above it, the upper part being now very bright; near s we have a section like the fourth figure, with the upper part very bright; and between s and the place of the least circle of aberration for the area, whose section is $bA'B'$, a section like the last figure, with an upper bright nucleus at b and the coma descending to a.

These appearances may be verified with an ordinary concave eye-glass; *using the light from a lamp passing through a hole in a card, for the luminous point, and receiving the light reflected* from one of the concave surfaces on a white screen. The appearances will be found to change with the obliquity, and to approximate to the primary and secondary focal lines and circle of least confusion, when the obliquity is considerable or the reflecting surface small compared with the focal length.

If we were to choose the most symmetrical small area for the focus, perhaps the middle figure, which corresponds to the circle of confusion, where the length equals the breadth, would be taken; but the eye is not satisfied that it is the nearest approximation to a focus, and prefers the last, with the bright nucleus, notwithstanding the lengthened coma.

Art. 13. Prop. *To find the length of* the coma *and of* the cometa or figure of oblique aberration *when a pencil of rays falls obliquely on a spherical mirror.*

Let $BbAA'B'$ be a section of the mirror by a plane through the axis of the mirror and the axis $A'q'oq$ of the oblique pencil; so that $A'B$ and $A'B'$ are the greatest and least distances from A' to the edge. Then $A'b$ being taken equal to $A'B'$, let the ordinate $bN = y$, and $BN' = y'$; let $Bq'a$ be the ray



reflected at B, cutting the axis in q', and forming the limit of the coma at a; s the intersection of the axis by the rays bs, $B's$; q the geometrical focus for rays reflected near A'; noa the figure of oblique aberration, so that no is the radius of the least circle of aberration for the area on the face of the mirror, whose diameter is $bA'B'$, and oa is the length of the coma.

From Art. 9 $$no=\frac{y^3v_1}{4r}\left(\frac{1}{r}-\frac{1}{u}\right)^2 \quad \ldots\ldots\ldots \quad (1)$$

From the similar triangles $q'oa$, $q'N'B$ we have

$$\frac{oa}{q'o}=\frac{BN'}{q'N'}=\frac{y'}{v_1} \text{ nearly,}$$

$$\text{or } oa=\frac{y'}{v_1}q'o \quad \ldots\ldots\ldots\ldots \quad (2)$$

also by Art. 9 $$qo=\frac{3}{4}qs$$

and $$\frac{qs}{qq'}=\frac{y^2}{y'^2}$$

$$\therefore q'o=qq'-qo=qq'-\frac{3}{4}qs$$

$$=qq'\left(1-\frac{3}{4}\frac{y^2}{y'^2}\right)$$

Substituting this value in (2) and the value of qq' from Art. 7, we have the length of the coma

$$=oa=\frac{y'^2v_1{}^2}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(1-\frac{3}{4}\frac{y^2}{y'^2}\right)\frac{y'}{v_1}$$

$$=\frac{y'^3v_1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(1-\frac{3}{4}\frac{y^2}{y'^2}\right)$$

If $y=y'$ or the obliquity vanishes, this expression becomes

$$\frac{y'^3v_1}{4r}\left(\frac{1}{r}-\frac{1}{u}\right)^2$$

the radius of the least circle of aberration found in Art. 9.

As the ratio $\frac{y}{y'}$ becomes smaller, the quantity to be subtracted in the last factor becomes rapidly less, and the brightness of the nucleus rapidly diminishing, the magnitude of the coma becomes rapidly increased, until the axis of the pencil falls on the edge of the mirror or A' coincides with B' when its value becomes, since then $y=0$,

$$\frac{y'^3 v_1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2$$

and we still consider the mirror of such aperture that the longitudinal aberration varies as y^2 nearly.

For the value of *noa*

we have $\quad no+oa=\dfrac{v_1}{r}\left(\dfrac{1}{r}-\dfrac{1}{u}\right)^2\left\{y'^3\left(1-\dfrac{3}{4}\dfrac{y^2}{y'^2}\right)+\dfrac{y^3}{4}\right\}$

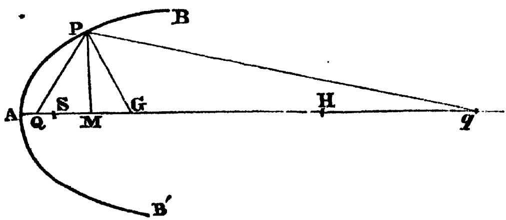
= the extreme length of the figure of oblique aberration.

The above discussion applies to the images formed by the object-mirrors of reflecting telescopes of the Newtonian, Gregorian and Cassegranian forms; the points in the images out of the axis being affected with oblique aberration when the mirror is spherical; and the obliquity of the pencils forming these images is never very great. See Chap. ix, PART I.

When a pencil from the object-mirror falls obliquely on the small mirror of the Gregorian or Cassegranian telescopes, we have the section perpendicular to the common axis of the mirrors a circle very nearly, because the aperture of the large mirror is so; but as the obliquity is always small, we may consider the form as a right cone; and this conical pencil falls entire on the small mirror, so that the circumstances are only different to those of the reflexion at the large mirror in the obliquity of the second reflexion being greater than that of the first.

ART. 14. PROP. *To find the aberration of* an ellipsoidal mirror *for a direct pencil; when the conjugate foci of the incident and reflected rays do not coincide with the foci of the mirror.*

Let BAB' be the ellipse, which is a section of the mirror, of which the foci are S and H. Taking a case



similar to that of the speculum of the reflecting microscope (PART I. page 140), let Q be the focus of the incident rays, $AQSHq$ the axis of the mirror, QP any ray reflected in Pq; and let PG be the normal at P.

Let $AQ=u$, $Aq=v$, $AM=x$, $PM=y$, and the equation of the ellipse $y^2=\frac{b^2}{a^2}(2\,ax-x^2)$; also, according to the ordinary notation, $e^2=1-\frac{b^2}{a^2}$.

Then the equation of the normal at P being

$$y-y'=-\frac{a^2y}{b^2\,(a-x)}(x-x')$$

at G we have $y'=0$, and $x'=AG=\frac{b^2}{a}+e^2x$.

Since the normal bisects the angle QPq we have as before

$$\frac{qG}{qP}=\frac{QG}{QP}$$

and proceeding as in Art. 7,

$$\frac{1}{Aq}=\frac{2}{AG}-\frac{1}{AQ}+2x\left(\frac{1}{AQ}-\frac{1}{AG}\right)^2$$

or $\frac{1}{v}=\frac{2}{\frac{b^2}{a}+e^2x}-\frac{1}{u}+2x\left(\frac{1}{u}-\frac{1}{\frac{b^2}{a}+e^2x}\right)^2$

or $\frac{1}{v}=\frac{2a}{b^2}-\frac{2e^2a^2x}{b^4}-\frac{1}{u}+2x\left(\frac{1}{u}-\frac{a}{b^2}\right)^2$ nearly, taking only terms with the first power of x;

or $\frac{1}{v}=\frac{2a}{b^2}-\frac{1}{u}+2x\left\{\left(\frac{1}{u}-\frac{a}{b^2}\right)^2-\frac{e^2a^2}{b^4}\right\}$

where the last term is the small variation of $\frac{1}{v}$, or $\delta\left(\frac{1}{v}\right)$

and $\qquad\delta v=\text{the aberration}=-v_1v\,.\,\delta\left(\frac{1}{v}\right)$

$$=-v_1^2 2x\left\{\left(\frac{1}{u}-\frac{a}{b^2}\right)^2-\frac{e^2a^2}{b^4}\right\} \quad \text{nearly,}$$

where v_1 is the first approximate value of v.

If we put $u=AS=a\ (1-e)=\dfrac{a(1-e^2)}{1+e}=\dfrac{b^2}{a(1+e)}$ we have the coefficient of $-v_1^2 2x$, thus

$$\left(\frac{a(1+e)}{b^2}-\frac{a}{b^2}\right)^2-\frac{e^2a^2}{b^4}=0$$

or the aberration vanishes, as was found in Art. 13, PART I.

If $e=0$ the ellipse becomes a circle, and the result becomes identical with the corresponding one of Art. 7. We see that the aberration, vanishing when Q and q coincide respectively with S and H, becomes greater as the term $\left(\dfrac{1}{u}-\dfrac{a}{b^2}\right)^2$ differs more from $\dfrac{e^2a^2}{b^4}$, and changes sign as Q passes from one side of S to the other.

If we use the approximate value, $x=\dfrac{ay^2}{2b^2}$, we have

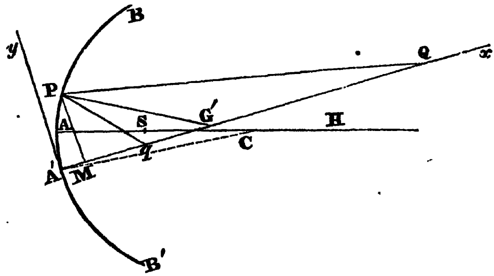
$$\delta v=-\frac{v_1^2ay^2}{b^2}\left\{\left(\frac{1}{u}-\frac{a}{b^2}\right)^2-\frac{e^2a^2}{b^4}\right\}$$

which we may employ to find the least circle of aberration, as in Art. 9.

ART. 15. PROP. *To investigate an expression for the aberration of a ray, of a pencil falling* obliquely *upon a concave ellipsoidal mirror, in the principal section.*

Let $BAA'B'$ be the elliptic section of the mirror in the plane of incidence of the ray QP; S and H being the foci of the ellipse and Q being the focus of the incident rays; also QA' the ray which falls perpendicularly on the mirror and is taken for the axis of the pencil.

Let PG' be the normal at P, meeting $A'Q$ in G'; and Pq the reflected ray meeting the same line in q.



Referring the ellipse to the tangent $A'y$ and normal $A'Qx$, as axes of co-ordinates, let $AM=x$, $PM=y$ be the co-ordinates of P; and C being the center of the ellipse, let α=angle $CA'G'$, between the semi-diameter $A'C$ and normal $A'G'$; then transforming the equation of the ellipse referred to $A'C$ and the conjugate diameter as axes of co-ordinates, to those in the figure we have the equation of the ellipse

$$\frac{(y+x\tan.\alpha)^2}{b'^2}=\frac{2x\sec.\alpha}{a'}-\frac{x^2\sec.^2\alpha}{a'^2}\;.\;.\;.\;.\;.\;(1)$$

where $a'=A'C$, and b' is the semi-diameter conjugate to it; these can be found in the given ellipse when the point A' is given; also if x_1y_1 are the co-ordinates of A' when the ellipse is referred to the center and principal axes, or when $\frac{y_1^2}{b^2}+\frac{x_1^2}{a^2}=1$ we have

$$\tan.\alpha=\tan.CA'Q=\tan(\angle\text{ between }A'Q\text{ and }AC-\angle A'CA)$$

$$=\frac{-\frac{dx_1}{dy_1}-\frac{y_1}{x_1}}{1-\frac{dx_1}{dy_1}\cdot\frac{y_1}{x_1}}$$

$$=\frac{\left(\frac{a^2}{b^2}-1\right)\frac{y_1}{x_1}}{1+\frac{a^2y_1^2}{b^2x_1^2}}$$

$$=\frac{e^2x_1y_1}{b^2}$$

Since PG' bisects the angle QPq, we have as before

$$\frac{qG'}{qP}=\frac{QG'}{QP}$$

and putting $A'q=v$, $A'Q=u$, and $A'G'=s$ a quantity varying with the position of P, of which the value is found below, we have as in the previous propositions

$$\begin{aligned}\frac{1}{v}&=\frac{2}{s}-\frac{1}{u}+2x\left(\frac{1}{s}-\frac{1}{u}\right)^2\\ &=\frac{2}{s}-\frac{1}{u}+\frac{y^2}{s}\left(\frac{1}{s}-\frac{1}{u}\right)^2 \quad \ldots\ldots \quad (2)\end{aligned}$$

To find s we have

$$\begin{aligned}s&=A'G'=A'M+MG'\\ &=x+y\frac{dy}{dx}\end{aligned}$$

Since α and y are small, and x very small, we may use from (1) the approximate value $x=\dfrac{y^2a'}{2b'^2 \sec.\alpha}+\dfrac{y^3a'^2 \tan.\alpha}{2b'^4 \sec.^2\alpha}$; and rejecting terms with powers of y above the second, we find

$$s=\frac{b^2 \sec.\alpha}{a'}-\frac{3}{2}y\tan.\alpha+\frac{y^2a'}{2b'^2\sec.\alpha}\left\{1-\frac{b'^2\sec.^2\alpha}{a'^2}-\frac{1}{2}\tan.^2\alpha\right\}$$

whence

$$\frac{1}{s}=\frac{a'}{b'^2\sec.\alpha}+\frac{3}{2}\frac{ya'^2\tan.\alpha}{b'^4\sec.^2\alpha}-\frac{y^2a'^3}{2b'^6\sec.^3\alpha}\left\{1-\frac{b'^2\sec.^2\alpha}{a'^2}-5\tan.^2\alpha\right\}$$

for substitution in the small term, we have the approximate value

$$\frac{1}{s}=\frac{a'}{b'^2\sec.\alpha}\text{ nearly;}$$

therefore the equation (2) becomes

$$\begin{aligned}\frac{1}{v}&=\frac{2a'}{b'^2\sec.\alpha}+\frac{3ya'^2\tan.\alpha}{b'^4\sec.^2\alpha}-\frac{y^2a'^3}{b'^6\sec.^3\alpha}\left\{1-\frac{b'^2\sec.^2\alpha}{a'^2}-5\tan.^2\alpha\right\}-\frac{1}{u}\\ &\qquad+\frac{y^2a'}{b'^2\sec.\alpha}\left(\frac{a'}{b'^2\sec.\alpha}-\frac{1}{u}\right)^2\\ &=\frac{2a'}{b'^2\sec.\alpha}-\frac{1}{u}+\frac{3ya'^2\tan.\alpha}{b'^4\sec.^2\alpha}+\frac{y^2a'}{b'^2\sec.\alpha}\left\{\left(\frac{a'}{b'^2\sec.\alpha}-\frac{1}{u}\right)^2-\frac{a'^2}{b'^4\sec.^2\alpha}\left(1-\frac{b'^2\sec.^2\alpha}{a'^2}-5\tan.^2\alpha\right)\right\}\end{aligned}$$

If we put v_1 the first approximate value of v, we have the aberration

$$=v_1 - v$$

$$=v_1 v\left(\frac{1}{v}-\frac{1}{v_1}\right)$$

When Q is so distant that $\frac{1}{u}$ may be neglected, we have the aberration

$$=\frac{b'^4 \sec.^2 \alpha}{4a'^2}\left(1-\frac{3}{2}\frac{y a' \tan.\alpha}{b'^2 \sec.\alpha}\right)\left\{\frac{3 y a'^2 \tan.\alpha}{b'^4 \sec.^2\alpha}+\frac{y^2 a'^3}{b'^6 \sec.^3\alpha}\left(\frac{b'^2 \sec.^2\alpha}{a'^2}+5\tan.^2\alpha\right)\right\}$$

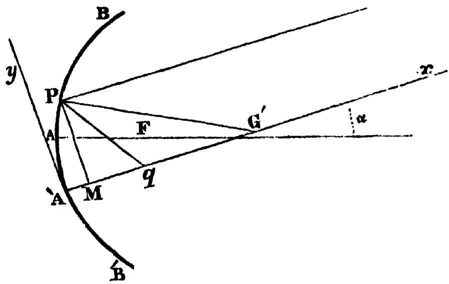
$$=\frac{3}{4} y \tan.\alpha+\frac{y^2 a'}{4b'^2 \sec.\alpha}\left\{\frac{b'^2 \sec.\alpha}{a'^2}+\frac{\tan.^2\alpha}{2}\right\}$$

We see that the aberration depends partly on a term with y, and partly on one with y^2, and is therefore not symmetrical for all parts of the pencil; the term with y has, however, the small quantity $\tan.\alpha$ a multiplier, and is therefore very small.

In the expression for $\frac{1}{v}$, the term with y^2 has a positive and negative part, and is very small when Q is near H or S. When we make $\alpha=0$, we have the result of the last proposition, Q being then in the axis of the mirror; and when further we make $a'=b'$, we have the results of Art. 7 for a spherical mirror; also if we make the major axis of the ellipse and u infinite, the results, agree with those of the next proposition, for a pencil of parallel rays falling obliquely on a parabolic mirror: so that the above results may be considered as the most general that we can take.

Art. 16. Prop. *To find the aberration in the principal section for a pencil of parallel rays falling* obliquely *on a paraboloidal mirror.*

Let AF be the axis of the paraboloidal mirror $BAA'B'$; and let $A'G'x$ be the ray of the oblique pencil which falls perpendicularly on the mirror: another of the parallel rays in the plane of



the figure, or principal section, being incident at P; let Pq be the reflected ray. If α be the angle which the ray $A'x$ makes with the axis of the mirror AF, and we refer the parabola $BA'B'$ to the normal $A'x$, and the tangent $A'y$ as axes of co-ordinates, we have the equation of the parabola

$$(y+x\tan.\alpha)^2=4m'.x\sec.\alpha$$

where m' is the distance of A' from F, the focus of the parabola.

Let PG' be the normal at P; since it makes equal angles with the incident and reflected rays, we have the triangle $PG'q$ isosceles, and $Pq=qG'$. Let $A'q=v$.

To find $A'G'$, let $A'M=x$, $PM=y$ be the co-ordinates of P; we have

$$A'G'=x+y\frac{dy}{dx}$$
$$=2m'\sec.\alpha-\frac{3}{2}y\tan.\alpha+\frac{y^2}{4m'\sec.\alpha}\left(1-\frac{\tan^2\alpha}{2}\right)$$

To find Pq, we have,

$$Pq=\sqrt{\overline{Mq}^2+\overline{PM}^2}$$
$$=\sqrt{(v-x)^2+y^2}$$
$$=v\sqrt{1-\frac{2x}{v}+\frac{y^2}{v^2}}\text{ nearly}$$
$$=v\left\{1-\frac{y^2}{2v}\left(\frac{1}{2m'\sec.\alpha}-\frac{1}{v}\right)\right\}\text{nearly}$$

and $\quad Pq=A'G'-A'q$

$$\therefore\quad v-\frac{y^2}{2}\left(\frac{1}{2m'\sec.\alpha}-\frac{1}{v}\right)=2m'\sec.\alpha-\frac{3}{2}y\tan.\alpha+\frac{y^2}{4m'\sec.\alpha}\left(1-\frac{\tan.^2\alpha}{2}\right)-v$$

whence $v = m' \sec.\alpha - \frac{3}{4} y \tan.\alpha + \frac{y^2}{4}\left\{\frac{1}{2m' \sec.\alpha}\left(1 - \frac{\tan.^2\alpha}{2}\right) + \left(\frac{1}{2m' \sec.\alpha} - \frac{1}{v}\right)\right\}$

and using the approximate value $v = m' \sec.\ \alpha$ in the small term, we have,

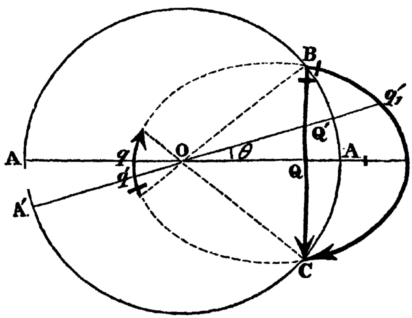
$$v = m' \sec.\ \alpha - \frac{3}{4} y\ \tan.\ \alpha - \frac{y^2 \tan.^2 \alpha}{16\, m' \sec.\ \alpha}.$$

Here the two latter terms are the aberration, which we see is always exceedingly small in the parabolic mirror of a telescope, where tan. α is always very small, as well as y.

From this discussion we learn that even in the oblique pencils, the parabolic has a great advantage over the spherical mirror in a reflecting telescope. The rays reflected in other planes of oblique pencils, will not meet the normal rays; but the error will in all actual cases be exceedingly small.

Art. 17. Prop. *To find the form of the image of a straight line, given by small direct pencils reflected by a concave spherical mirror.*

Let BQC be the straight line, and $BACA_1$ the section of the sphere of the mirror by a plane through the straight line and center O. Then the foci for small direct pencils will lie all in this plane. It is clear that the part of the straight line from which rays can fall on the face of the mirror is the part within the circular section, when it has a position as in the figure; and there will be two images, one real by the mirror BAC, and the other virtual by BA_1C.



Let $AOQA_1$ be the diameter perpendicular to BQC, and let q be the real image of Q by the mirror BAC; q_1 the virtual one by BA_1C.

If we take any other point as Q', we shall have the two images q' and q'_1 as in the figure, in the line $A'OQ'$; and to find the loci, or images of the various points of the given straight line, it is now more convenient to refer the positions of the conjugate foci to the center O. Thus, let $Q'O=p$, $q'O=q$ and $AO=r$; we have from Art. 19, PART I.

$$\frac{q'A'}{q'O}=\frac{Q'A'}{Q'O}$$

or
$$\frac{r-q}{q}=\frac{r+p}{p}$$

$$\therefore \quad \frac{1}{q}=\frac{2}{r}+\frac{1}{p}$$

Let the perpendicular distance $OQ=m$, and $\angle Q'OQ=\theta$,

we have $\qquad p=m \sec. \theta$

and $\qquad \dfrac{1}{q}=\dfrac{2}{r}+\dfrac{1}{m \sec. \theta}$

or $\qquad q=\dfrac{rm}{2m+r\cos. \theta}=\dfrac{\frac{r}{2}}{1+\frac{r}{2m}\cos. \theta}$

Comparing with the polar equation of a conic section

$$\rho=\frac{l}{1+e\cos. \theta}$$

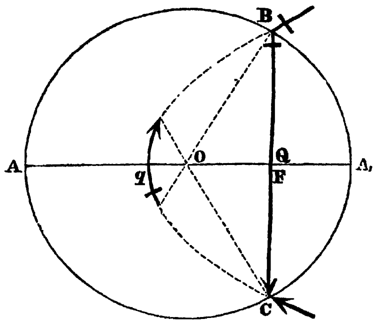
we see that the image is an ellipse when $\frac{r}{2}$ is less than m

. . . . a parabola . . $\frac{r}{2}$ is equal to m

. . . . an hyperbola . $\frac{r}{2}$ is greater than m

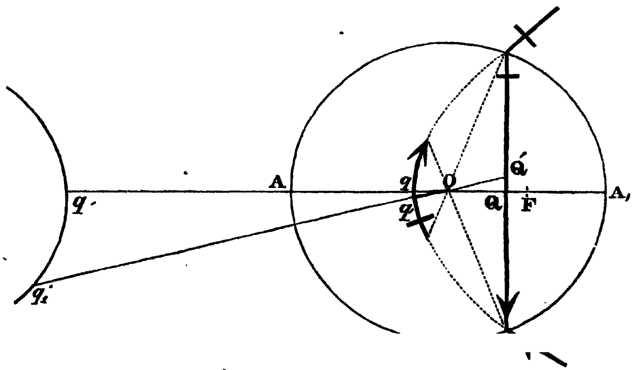
and in all cases the half latus rectum l equals $\frac{r}{2}$ the focal length of the mirror; and the center O of the mirror is the focus of the conic section. As a particular case of the ellipse, the image

becomes a circular arc when m is infinite, and $e=\frac{r}{2m}=0$; or the image of a very distant object, as a heavenly body, is formed in a spherical surface concentric with the mirror, and of radius the focal distance; for although the visible discs of the sun and moon are spherical, yet the relative distances of different points in them are so nearly the same, that the images of them formed by a small telescopic mirror will be indefinitely nearly as stated. This is a point of consequence in investigating the best form of an eye-piece for the Newtonian telescope.



The parabolic images take place as in the annexed figure.

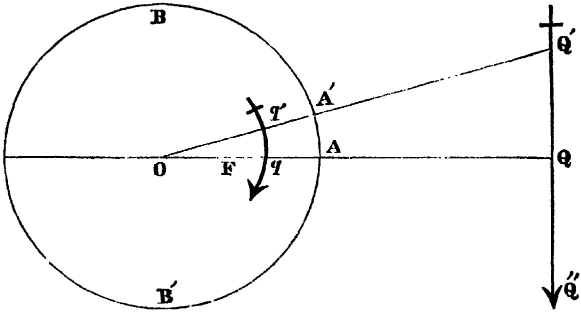
The hyperbolic images take place as in the annexed figure.



When the straight line is outside the circular section of the sphere of the concave mirror, there can evidently be only one image formed, which is an elliptic arc, as in the first figure, but of diminishing eccentricity as the distance m becomes greater.

ART. 18. PROP. *To find the image formed by small direct pencils, of a straight line placed before a* convex *spherical mirror.*

Let $Q'QQ''$ be the straight line, and $BA'AB'$ be the section of the sphere of the mirror; and OAQ the perpendicular from the



center O upon the line. Let q be the virtual focus conjugate to Q, and taking any other point Q' in the straight line, join O, Q' and let q' be the focus conjugate to Q'; also let $\angle Q'OQ=\theta$, $OQ=m$, $OQ'=p$, $Oq'=q$, $OA=r$, then as in Art. 20, PART I.

$$\frac{q'A'}{q'O}=\frac{Q'A'}{Q'O}$$

or
$$\frac{r-q}{q}=\frac{p-r}{p}$$

whence
$$\frac{1}{q}=\frac{2}{r}-\frac{1}{p}$$

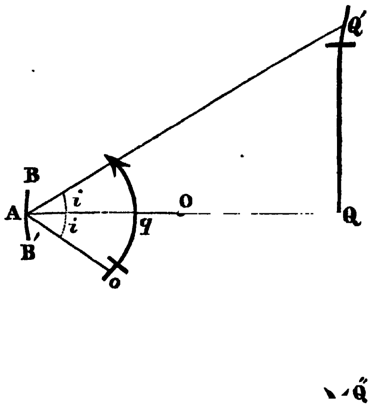
$$=\frac{2}{r}-\frac{1}{m\sec.\theta}$$

$$\therefore q=\frac{\frac{r}{2}}{1-\frac{r}{2m}\cos.\theta}$$

which is the polar equation of an ellipse, with $\frac{r}{2}=$ the half latus rectum, and O the further focus.

ART. 19. PROP. *To find the form of the image of a straight line, produced by* oblique *pencils, reflected at a small concave spherical mirror.*

We have now to consider the image as the locus of the circles of least confusion, in the obliquely reflected pencils.



Let BAB' be the small mirror, of which the center of curvature is O, and $Q'QQ''$ the straight line; let AOQ be the axis of the mirror perpendicular to $Q'QQ''$, and q the focus conjugate to Q.

If we take any other point, as Q', join A, Q', and draw Ao, making $\angle oAQ = \angle Q'AQ$, then the least circle of confusion for the reflected pencil will be at some point in this last line, as at o.

To find the form of the image, let $AQ=a$, $AO=r$, $Ao=v$, $AQ'=u$, $\angle Q'AQ=i$; then $u=a$ sec. i, and by Art. 11, we have

$$\frac{1}{v_1}=\frac{2\text{ sec. } i}{r}-\frac{1}{u}=\frac{2\left(1+\frac{i^2}{2}\right)}{r}-\frac{1-\frac{i^2}{2}}{a} \text{ nearly}$$

$$\frac{1}{v_2}=\frac{2\text{ cos. } i}{r}-\frac{1}{u}=\frac{2\left(1-\frac{i^2}{2}\right)}{r}-\frac{1-\frac{i^2}{2}}{a} \quad . \quad . \quad .$$

adding we have

$$\frac{1}{v_1}+\frac{1}{v_2}=\frac{4}{r}-\frac{2}{a}+\frac{i^2}{a}=2\left(\frac{2a-r}{ra}\right)\left(1+\frac{i^2 r}{2(2a-r)}\right)$$

and $\frac{1}{v_1}\cdot\frac{1}{v_2}=\left\{\frac{2}{r}-\frac{1}{a}+\frac{i^2}{2}\left(\frac{2}{r}+\frac{1}{a}\right)\right\}\times\left\{\frac{2}{r}-\frac{1}{a}-\frac{i^2}{2}\left(\frac{2}{r}-\frac{1}{a}\right)\right\}$

$$=\left(\frac{2}{r}-\frac{1}{a}\right)^2+\frac{i^2}{a}\left(\frac{2}{r}-\frac{1}{a}\right)$$

$$=\left(\frac{2\,a-r}{ra}\right)^2\left\{1+\frac{i^2\,r}{2\,a-r}\right\}$$

or $$v_1\,v_2=\left(\frac{ra}{2\,a-r}\right)^2\left\{1-\frac{i^2\,r}{2\,a-r}\right\}$$

But since the least circle of confusion, by Art. 12, is nearly equally distant from the primary and secondary focal lines, we have

$$Ao=v=\frac{v_1+v_2}{2}$$

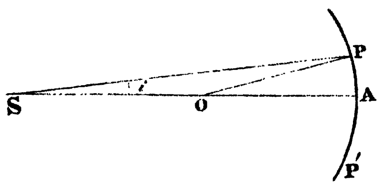
$$=\frac{v_1 v_2}{2}\left(\frac{1}{v_1}+\frac{1}{v_2}\right)$$

$$=\left(\frac{ra}{2\,a-r}\right)^2\left\{1-\frac{i^2r}{2\,a-r}\right\}\times\left(\frac{2\,a-r}{ra}\right)\left\{1+\frac{i^2r}{2(2\,a-r)}\right\}$$

$$=\frac{ra}{2\,a-r}\left(1-\frac{i^2r}{2\,(2a-r)}\right)$$

$$=\frac{ra}{2\,a-r}-\frac{i^2\,.\,r^2a}{2\,(2\,a-r)^2}\quad\ldots\ldots\ldots\ldots\ldots\quad(1)$$

We may compare this expression with the *approximate* polar equation to a circular arc; obtained by putting as before, cos. $i=1-\frac{i^2}{2}$ nearly, in the value of OP^2 from the triangle SOP in the figure; where O is the center of the circular arc PAP', and S the pole.



Let $OA=r'$, $SA=a'$, $SP=\rho$, $\angle PSA=i$,

then $$r'^2=\rho^2+(a'-r')^2-2\rho(a'-r')\cos.\ i,$$

from which we find the approximate equation

$$\rho=a'-i^2\,.\,\frac{a'\,(a'-r')}{2r'}$$

When the circle is convex towards S, and therefore A the nearest point, we find

$$\rho = a' + i^2 \cdot \frac{a'(a'+r')}{2r'}$$

If $\rho = v$ in the above equation (1) we have

$$a' = \frac{ra}{2a-r} = Aq \text{ in the first figure}$$

$$\frac{a'(a'-r')}{r'} = \frac{r^2 a}{(2a-r)^2} = \frac{a'^2}{a}$$

$$\therefore \; a(a'-r') = r'a'$$

$$r' = \frac{aa'}{a'+a}$$

$$= \frac{ra^2}{(2a-r)\left(\frac{ra}{2a-r}+a\right)}$$

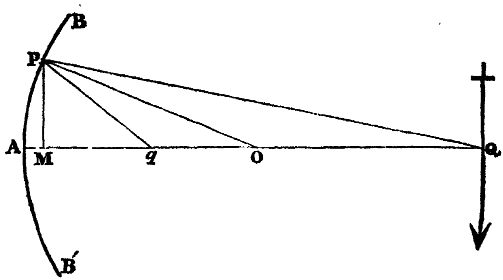
$$= \frac{r}{2}$$

or the image is approximately a circular arc with radius $\frac{r}{2}$.

We see that an image formed in this manner is *concave* towards the mirror, whereas from Art. 17, the image formed by small direct pencils is *convex* towards it; neither, however, of these two modes is that in which the image formed by the object-mirror of the Newtonian, Gregorian, or Cassegranian telescope arises, for it must more strictly be considered as the locus of the figure of aberration, as found in the next proposition.

ART. 20. PROP. *To find the form of the image of a straight line produced by a* concave *spherical mirror, when it is considered the locus of the cometa of oblique aberration.*

As in Art. 17, we have to refer the image to the center of curvature of the mirror, and now require expressions for the direct and oblique aberrations in terms of the polar co-ordinates.



Let AOQ be the axis of the mirror BAB', meeting the straight line perpendicularly at Q; and let O be the center of curvature of the mirror. Let any ray QP be reflected in Pq, and let $AO=r$, $QO=p$, $qO=q$; also $AM=x$, $PM=y$, the co-ordinates of P.

Then as before, we have $\dfrac{qP}{qO}=\dfrac{QP}{QO}$ . . . . (1)

and $$QP^2=QO^2+OP^2+2OM\,.\,OQ$$
$$=p^2+r^2+2(r-x)\,p$$
$$=(p+r)^2-2px$$
$$\therefore\quad QP=(p+r)\sqrt{1-\frac{2px}{(p+r)^2}}$$
$$=p+r-\frac{px}{p+r}\text{ nearly}$$
$$=p+r-\frac{y^2p}{2r\,(p+r)}$$

again $$qP^2=qO^2+OP^2-2OM\,.\,Oq$$
$$=q^2+r^2-2(r-x)q$$
$$=(r-q)^2+2qx$$
$$\therefore\quad qP=(r-q)\sqrt{1+\frac{2qx}{(r-q)^2}}$$
$$=r-q+\frac{qx}{r-q}\text{ nearly}$$
$$=r-q+\frac{y^2q}{2r(r-q)}$$

Substituting these values in (1) we have

$$\frac{r-q+\dfrac{y^2q}{2r(r-q)}}{q}=\frac{p+r-\dfrac{y^2p}{2r(p+r)}}{p}$$

or
$$\frac{1}{q}=\frac{2}{r}+\frac{1}{p}-\frac{y^2}{2r^2}\left\{\frac{1}{p+r}+\frac{1}{r-q}\right\}$$

substituting the first approximate value of q in the small term, we find,

$$\frac{1}{q}=\frac{2}{r}+\frac{1}{p}-\frac{y^2}{r^3}$$

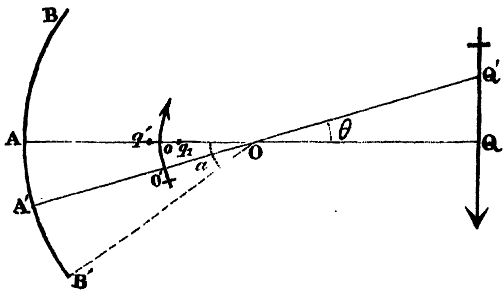
$$=\frac{2p+r}{rp}\left\{1-\frac{y^2p}{r^2(2p+r)}\right\}$$

whence
$$q=\frac{rp}{2p+r}+\frac{y^2p^2}{r(2p+r)^2}\ \ldots\ldots\ldots\ (2)$$

where the last term is the aberration of a ray incident on the mirror at a point distant y from the axis.

Let Q' be any point in the line; and let $Q'OA'$ be the ray falling perpendicularly on the mirror, which is taken for the axis of the oblique pencil, making $\angle\, Q'OQ=\theta$.

If q_1 be the first approximate focus of the direct pencil; and q_1q' the aberration of the extreme ray, then the place of o, the least circle of aberration, is such that $q_1o=\frac{3}{4}q_1q'$ by Art. 9. Also, if o' be the place of the cometa of oblique aberration in the ray $Q'OA'$, it will be a point in the image.



Let α be the angle BOA or $B'OA$, of the semi-aperture of the

mirror at O, we have $y=r\alpha$ nearly for the extreme ray; and $A'B'=r(\alpha-\theta)$ nearly.

If we put now $OQ=m$, $OQ'=p$, $Oo'=q$, we have

$$p=m \sec.\theta=m\left(1+\frac{\theta^2}{2}\right) \text{ nearly.}$$

Now, by Art. 13, and substituting the corresponding quantities in (2) we have,

$$q=Oo'=\frac{r\,.\,OQ'}{2OQ'+r}+\frac{3}{4}\frac{A'B'^2.OQ'^2}{r\,(2OQ'+r)^2}$$

$$=\frac{r\,.\,m\sec.\theta}{2m\sec.\theta+r}+\frac{3}{4}\frac{r^2(\alpha-\theta)^2m^2\sec.^2\theta}{r\,(2m\sec.\theta+r)^2}$$

$$=\frac{rm\left(1+\frac{\theta^2}{2}\right)}{2m\left(1+\frac{\theta^2}{2}\right)+r}+\frac{3}{4}\frac{r\,(\alpha-\theta)^2m^2\left(1+\frac{\theta^2}{2}\right)^2}{\left(2m\left(1+\frac{\theta^2}{2}\right)+r\right)^2}$$

$$=\frac{rm}{2m+r}+\frac{\theta^2r^2m}{2(2m+r)^2}+\frac{3}{4}\frac{r(\alpha-\theta)^2m^2}{(2m+r)^2}\left\{1+\frac{\theta^2r}{2m+r}\right\}$$

If we take α small as in telescopic mirrors generally, and θ for the central parts of the field of view, small in comparison with α, we have the principal terms in the value of q thus

$$q=\frac{rm}{2m+r}+\frac{3}{4}\frac{r\alpha^2m^2}{(2m+r)^2}-\frac{3}{2}\frac{r\alpha\theta m^2}{(2m+r)^2}$$

which being of the form

$$q=a-b\theta$$

where b is small, is the equation of an arc of a spiral: also α and θ changing sign together for points on the other side of the axis AQ, the whole image will be composed of two such arcs, meeting at a very obtuse angle at o; with the convexity turned towards the mirror.

When, in the next figure, we take q for the luminous point, of

which the image is at Q, we have from the expressions before found

$$\frac{1}{p}=\frac{1}{q}-\frac{2}{r}+\frac{y^2}{2r^2}\left(\frac{1}{p+r}+\frac{1}{r-q}\right)$$

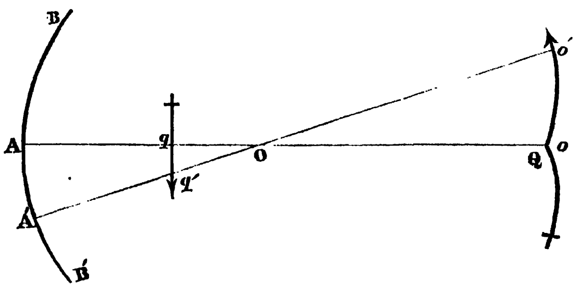
substituting the first approximate value of p in the small term, we find

$$\frac{1}{p}=\frac{1}{q}-\frac{2}{r}+\frac{y^2}{r^3}$$

whence
$$p=\frac{rq}{r-2q}-\frac{y^2q^2}{r(r-2q)^2}$$

where the last term is the aberration, which the negative sign shews, is measured towards the mirror.

Taking now q' any point in the straight line, let $A'q'Oo'$ be the axis of the oblique pencil; and $Oo'=p$, where o' is the place of the cometa of oblique aberration in the figure.



Then, as before, if $Oq=m$, and $\angle\, q'Oq=\theta$,

$$p=Oo'=\frac{r\,.\,Oq'}{r-2Oq'}-\frac{3}{4}\cdot\frac{A'B'^2\,.\,Oq'^2}{r(r-2Oq')^2}$$

$$=\frac{r\,.\,Oq\,.\,\sec.\,\theta}{r-2Oq\,.\,\sec.\,\theta}-\frac{3}{4}\cdot\frac{A'B'^2\,.\,Oq^2\,.\,\sec.^2\theta}{r(r-2Oq\,.\,\sec.\,\theta)^2}$$

$$=\frac{rm\left(1+\frac{\theta^2}{2}\right)}{r-2m\left(1+\frac{\theta^2}{2}\right)}-\frac{3}{4}\cdot\frac{r^2(\alpha-\theta)^2m^2\,(1+\theta^2)}{r\left[r-2m\left(1+\frac{\theta^2}{2}\right)\right]^2}\quad\text{nearly}$$

$$=\frac{rm}{r-2m}+\frac{\theta^2r^2m}{2(r-2m)^2}-\frac{3}{4}\cdot\frac{rm^2}{(r-2m)^2}(\alpha^2-2\alpha\theta+\theta^2)\left\{1+\frac{\theta^2r}{r-2m}\right\}$$

or retaining only the principal terms as before, we have

$$p=\frac{rm}{r-2m}-\frac{3}{4}\cdot\frac{rm^2\alpha^2}{(r-2m)^2}+\frac{3}{2}\frac{rm^2\alpha\theta}{(r-2m)^2}$$

which is of the form

$$p=a+b\theta$$

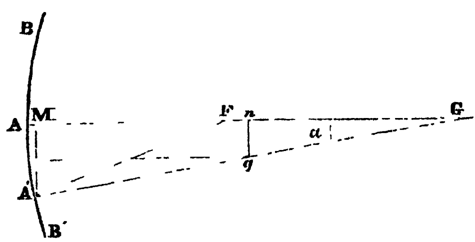
or the image of the straight line consists of two arcs of spirals again, as in the figure, meeting in a very obtuse angle at o.

The same method applies to finding the form of the image produced by a spherical mirror when the object is of any other given form, as a circular arc or any other curve line; and thus to finding the form of the second image in the Gregorian and Cassegranian telescopes, supposing the mirrors spherical.

In each of the last four articles we see that the image of a straight line is curved, and that *small* corresponding parts of the object and image are similar to each other, but the ratio of their magnitudes is variable; so that different parts of the image are differently magnified or diminished, or the image is distorted. For the small portion of the image, near the axis, which is seen in reflecting telescopes and microscopes, the distortion is not sensible, and the curvature requires to be considered in the construction of the eye-piece.

ART. 21. PROP. *To find the form of the image of a distant object, given by a parabolic mirror.*

Referring to Art 16. Let BAB' be the parabolic section of the mirror; $A'G$ the normal at A', and the axis of an oblique pencil, making $\angle A'GA=\alpha$ with the axis of the mirror: then the focus for the direct pencil being



at the focus of the parabola F, that of the oblique pencil will be at some point q, of which the position is determined by the result of Art. 16, as follows,

$$A'q=v=m' \sec. \alpha - \frac{3}{4} y \tan. \alpha - \frac{y^2 \tan.^2 \alpha}{16m' \sec. \alpha}$$

where $m'=A'F$, $y=$ the distance of any one of the parallel rays from the axis of the oblique pencil $A'G$.

The aberration vanishing with α, and changing sign with y, for points on different sides of the axis $A'G$, we may take the image as the locus of q determined by the expression

$$v=m' \sec. \alpha.$$

Let $AM=x'$, $A'M=y'$, be the co-ordinates of A', $An=x_1$, $qn=y_1$ those of q to origin A: and let $y^2=4mx$ be the equation of the parabola. By the property of the parabola we have the subnormal $MG=2m$.

$$\therefore \qquad \tan. \alpha = \frac{A'M}{MG} = \frac{y'}{2m}$$

Now
$$\begin{aligned} x_1 &= An = AM + Mn = x' + v \cos. \alpha = x' + m' \\ &= x' + A'F = x' + \sqrt{y'^2 + (m-x')^2} \\ &= x' + \sqrt{y'^2 + m^2 - 2mx'} \text{ nearly} \\ &= x' + m\sqrt{1 + \frac{y'^2}{2m^2}}. \\ &= x' + m + \frac{y'^2}{4m} \text{ nearly} \quad \ldots\ldots\ldots\ldots (1) \end{aligned}$$

again,
$$\begin{aligned} y_1 &= qn = A'M - v \sin. \alpha = y' - m' \tan. \alpha \\ &= y' - \frac{y'}{2m}\left(m + \frac{y'^2}{4m}\right) \\ &= \frac{y'}{2} \text{ neglecting } y'^3 \end{aligned}$$

$$\therefore \qquad y' = 2y_1 \quad \text{and substituting in (1)}$$

$$x' = x_1 - m - \frac{y_1^2}{m}$$

substituting these values in the equation of the parabola,

$$y'^2 = 4mx'$$

we have, $$4y_1^2 = 4m\left(x_1 - m - \frac{y_1^2}{m}\right)$$

$$\therefore \quad y_1^2 = \frac{m}{2}(x_1 - m)$$

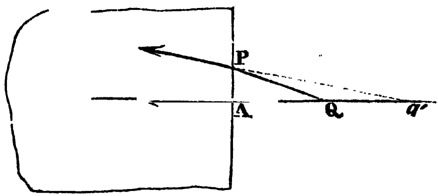
which is the equation of a parabola with its vertex at F, its convexity towards the mirror, and its latus rectum one-eighth of that of the mirror; so that we may consider the small portion which forms the field of view in a Newtonian telescope with a parabolic mirror to be nearly plane, with a slight concavity towards the eye-glass.

CHAPTER III.

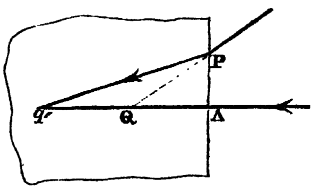
ON THE REFRACTION OF LIGHT AT PLANE AND CURVED SURFACES.

IN PART I. the propositions on the refraction of light were discussed only to first approximations, we have now to proceed to second approximations, with other higher propositions.

ART. 22. PROP. *To find the form of a small pencil of rays refracted directly at the plane surface of a medium to a second approximation.*



Let Q be the focus of the incident rays in the figures, for a diverging pencil in the upper one, and a converging pencil in the lower. Let QA be the ray falling perpendicularly on the surface and entering it without deviation; QP any other ray refracted in the direction of $q'P$.



Let $AQ=u$, $Aq'=u'$, $AP=y$,

$\angle PQA=i=$angle of incidence at P,

$\angle Pq'A=i'=$ . . . refraction at P,

we have in triangle QPq', as in Art. 28, PART I.,

$$\frac{Pq'}{PQ}=\frac{\sin. i}{\sin. i'}=\mu$$

or

$$\sqrt{u'^2+y^2}=\mu\sqrt{u^2+y^2}$$

$$u'\left(1+\frac{y^2}{2u'^2}\right)=\mu u\left(1+\frac{y^2}{2u^2}\right) \text{ nearly}$$

$$\therefore \quad u'=\mu u+\frac{y^2}{2}\left(\frac{\mu}{u}-\frac{1}{u'}\right)$$

$$=\mu u+\frac{y^2(\mu^2-1)}{2\mu u} \quad . \; . \; . \text{ by substituting for } u' \text{ its first}$$

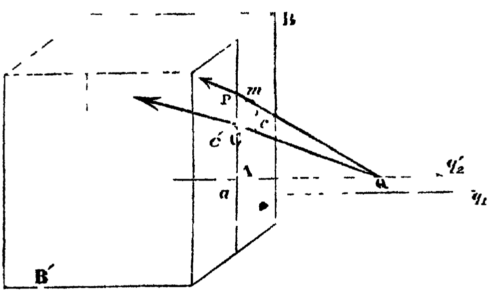
approximate value in the small term.

The last term is the aberration of the refracted ray, which varies as y^2, and the least circle of aberration will be found from the results of Art. 9.

If a conical pencil of rays be incident at a small obliquity, there will be oblique aberration, as in spherical mirrors, and the nearest approximation to a focus must be considered, as determined by the figure of oblique aberration, according to the results of Art. 13.

Art. 23. Prop. *To find the form of a small pencil of rays refracted* obliquely *at the plane surface of a medium.*

Let BB' be the refracting medium, Q the focus of the incident pencil, and QC any ray, which we may take for its axis. Draw QA perpendicular to the plane refracting surface of the medium; then QAC is the



plane of incidence, and $\angle CQA=i=$angle of incidence of QC: let the direction of the refracted ray meet QA in q'_2, then $\angle Cq'_2A=i'=$angle of refraction.

If we take QP a ray in the plane of incidence indefinitely near to QC, the refracted ray will have a direction Pq'_1 meeting Cq'_2 in

q'_1, which is the position of the line of *primary* foci, parallel to the refracting surface; being formed by the intersections of the rays refracted in other planes.

If with A for center, we describe a circular arc cCc' through C, every ray incident upon the refracting surface in this arc will be similarly situated with respect to the line QA, and the refracted rays will all meet it in the same point q'_2. The rays incident on the surface in other arcs concentric with cCc' will intersect the line QA in other points, and form the line of *secondary* foci.

To find the positions of q'_1 and q'_2, let $QC=u$, $q'_1C=u_1'$, $q'_2C=u'_2$; draw Cn a perpendicular on QP, and Cm another on q'_1P; then Pn, Pm are the increments of u and u'_1, and

$$\frac{du}{du'_1}=\frac{Pn}{Pm}=\frac{CP \sin. i}{CP \sin. i'} \ . \ . \ . \text{ ultimately}$$

$$=\mu \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (1)$$

Draw q'_1a perpendicular to the refracting surface, we have

$$q'_1a=u'_1 \cos. i', \quad \text{and} \quad QA=u \cos. i.$$

Now q'_1a and QA remain constant, whilst u, u'_1, i and i', vary in passing from the point C to P; therefore differentiating we have

$$0=\cos. i' \, du_1'-u'_1 \sin. i' \, . \, di'$$
$$0=\cos. i \, du \ -u \ \sin. i \, . \, di$$

whence

$$\frac{di}{di'}=\frac{\cos. i}{\cos. i'}\cdot\frac{u'_1}{u}\cdot\frac{du}{du'_1}\cdot\frac{\sin. i'}{\sin. i}$$

$$=\frac{\cos. i}{\cos. i'}\cdot\frac{u'_1}{u} \ . \ . \ . \ . \ . \ . \ . \ . \ . \text{ from (1)}$$

also differentiating $\sin. i=\mu \sin. i'$, we have

$$\frac{di}{di'}=\mu\frac{\cos. i'}{\cos. i}$$

and equating these values of $\frac{di}{di'}$

$$u'_1=\mu u \frac{\cos.^2 i'}{\cos.^2 i}$$

Again $AC = QC \sin. i = q'_2 C \sin. i'$

$$\therefore \quad u'_2 = u \frac{\sin. i}{\sin. i'}$$

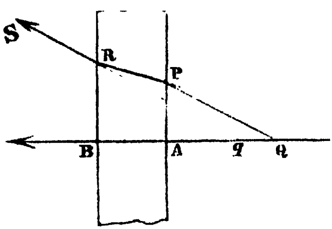
$$= \mu u$$

Between the focal lines there will be a circle of confusion, of which the magnitude and position may be determined as in the case of spherical mirrors.

Cor. If the incident pencil had been a converging one, we should have found the same result; like as in the previous proposition the two cases gave the same expression.

Art. 24. Prop. *To find the form of the emergent pencil to a second approximation when a small direct pencil has traversed a refracting plate of a medium.*

Let $PABR$ be the refracting plate, of which the thickness $AB = t$, let Q be the focus of the incident rays, and QAB the ray perpendicular to the plate and the axis of the pencil; let QP be any incident ray and $q'PR$ the direction of the refracted ray within the plate, also qRS that of the emergent ray, which is parallel to QP.



Let $QA = u$, $q'A = u'$, $qB = v$, $AP = y$, $RB = y'$

we have, $$\frac{y'}{y} = \frac{Bq'}{Aq'} \quad \text{or} \quad y' = y\left(\frac{u'+t}{u'}\right) = y\left(1 + \frac{t}{u'}\right)$$

From Art. 22,

$$u' = \mu u + \frac{y^2(\mu^2 - 1)}{2\mu u}$$

also, supposing the pencil to be incident in the contrary direction,

$$Bq' = u' + t = \mu v + \frac{y'^2(\mu^2-1)}{2\mu v}$$

$$= \mu u + \frac{y^2(\mu^2-1)}{2\mu u} + t$$

$$\therefore \qquad v = u + \frac{t}{\mu} - \frac{\mu^2-1}{2\mu^2}\left\{\frac{y'^2}{v} - \frac{y^2}{u}\right\}$$

When t is very small $y' = y$ nearly, and the aberrations at the two surfaces destroy each other. When t is small but not negligible

$$\frac{y'^2}{v} = \frac{y^2\left(1 + \frac{t}{u'}\right)^2}{u + \frac{t}{\mu}} \text{ nearly}$$

$$= \frac{y^2}{u}\left(1 + \frac{2t}{\mu u}\right)\left(1 - \frac{t}{\mu u}\right) \text{ nearly}$$

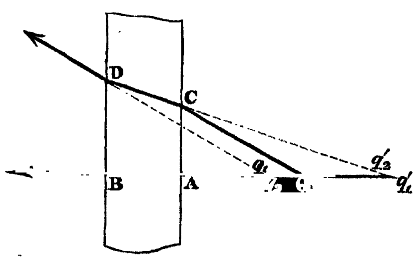
$$= \frac{y^2}{u}\left(1 + \frac{t}{\mu u}\right)$$

$$\therefore \qquad v = u + \frac{t}{\mu} - \frac{y^2(\mu^2-1)}{2\mu^2 u}\left(1 + \frac{t}{\mu u} - 1\right)$$

$$= u + \frac{t}{\mu} - \frac{y^2(\mu^2-1) \cdot t}{2\mu^3 u^2}$$

or the aberration depends on the thickness.

Art. 25. Prop. *To find the form of a small pencil transmitted* obliquely *through a refracting plate.*



Let QC be the axis of the small pencil diverging from Q, and falling obliquely on the plate; $q'_1\, q'_2$ the primary and secondary foci of the refracted pencil within the plate; $q_1\, q_2$ the primary and secondary foci of the emergent pencil respectively.

Let $QC=u$, $q'_1C=u'_1$, $q'_2C=u'_2$
$q_1D=v_1$, $q_2D=v_2$, $AB=t$,

then if QAB be perpendicular to the plate, the angle of incidence of $QC=\angle CQA=i$; and angle of refraction $Cq'_2A=i'$; also $DC=AB$ sec. $i'=t$ sec. i'.

At the first surface we have by Art. 23,

$$u'_1=\mu u\,\frac{\cos.^2 i'}{\cos.^2 i}$$

$$u'_2=\mu u$$

At the second surface, supposing the pencil incident in the contrary direction we have, by the Cor. Art. 23,

$$Dq'_1=\mu v_1\frac{\cos.^2 i'}{\cos.^2 i}$$

$$=u'_1+t\ \text{sec.}\ i'$$

$$=\mu u\frac{\cos.^2 i'}{\cos.^2 i}+t\ \text{sec.}\ i'$$

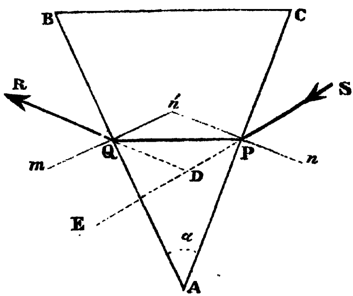
$$\therefore\qquad v_1=u+\frac{t\ \text{sec.}\ i'}{\mu}\cdot\frac{\cos.^2 i}{\cos.^2 i'}$$

Again
$$Dq'_2=\mu v_2$$
$$=u'_2+t\ \text{sec.}\ i'$$
$$=\mu u+t\ \text{sec.}\ i'$$
$$\therefore\qquad v_2=u+\frac{t\ \text{sec.}\ i'}{\mu}$$

we see that the confusion is a function of the thickness and the obliquity, and is very small for thin plates.

ART. 26. PROP. *To find the ray which passes through a given prism with the minimum deviation.*

Referring to Art. 33, PART I. Let α = the angle BAC of the prism, $i=\angle SPn=$ the angle of incidence at P, $i'=\angle n'PQ$ =angle of refraction at P; $e'=\angle n'QP=$ angle of incidence



on the second surface at Q, $e = \angle RQm$ = angle of emergence at Q.

$d = \angle QDE$ = the deviation.

We have as in the above-named article,

$$\alpha = i' + e' \quad \ldots\ldots\ldots \quad (1)$$
$$d = (i - i') + (e - e')$$
$$= i + e - \alpha \quad \ldots\ldots\ldots \quad (2)$$

We have also the relations sin. $i = \mu$ sin. i', sin. $e = \mu$ sin. e', from which we obtain the values of the differential coefficients $\frac{di}{di'}$, $\frac{de}{de'}$, and d is therefore a function of one [illegible] only, let this be taken i',

then from (1) $$\frac{de'}{di'} = -1$$

and from (2) $$\frac{d\,(d)}{di'} = \frac{di}{di'} + \frac{de}{de'} \cdot \frac{de'}{di'}$$

$$= \frac{\mu \cos. i'}{\cos. i} - \frac{\mu \cos. e'}{\cos. e}$$

$= 0$ for the minimum value of d; or $i' = e'$ and $i = e$, for this value.

To shew that these correspond to a *minimum* value of d, taking the second differential coefficient in respect of i', we have

$$\frac{d^2(d)}{di'^2} = \frac{d^2 i}{di'^2} + \frac{d^2 e}{de'^2} \cdot \left(\frac{de'}{di'}\right)^2$$

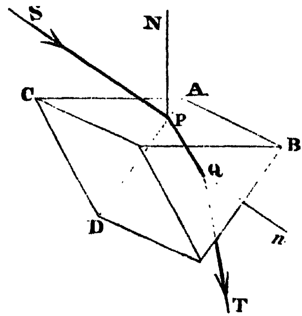
$$= \mu \left\{ \frac{\mu \cos.^2 i' \sin. i - \cos.^2 i \sin. i'}{\cos.^3 i} + \frac{\mu \cos.^2 e' \sin. e - \cos.^2 e \sin. e'}{\cos.^3 e} \right\}$$

or for the above found values

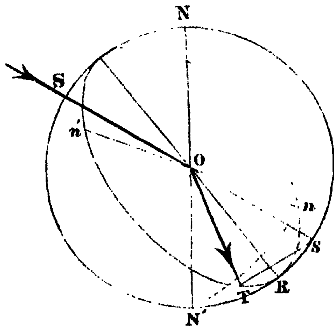
$$\frac{d^2(d)}{di'^2} = \frac{2\mu\,(\mu \cos.^2 i' \sin. i - \cos.^2 i \sin. i')}{\cos.^3 i}$$

which is positive, since always, sin. $i >$ sin. i' and cos. $i' >$ cos. i and the value of $d = 2\,i - \alpha$ is a minimum.

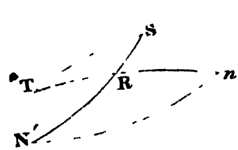
ART. 27. PROP. *To find the direction of a ray of light after refraction in any planes by a prism.*



Let AB be the *edge* of the prism, in the first figure, being the intersection of the plane faces CB and BD: let SP be a ray incident at P, which is then refracted in PQ, and emergent at Q in QT; so that the refractions at P and Q are in different planes; PN being the normal at P, Qn that at Q.



If a pencil of parallel rays falls upon the prism, the refracted pencil within the prism consists of parallel rays (Art. 25, PART I.), as also the emergent pencil; so that if we consider the circumstances of a ray falling indefinitely near the edge, we shall need to investigate only the directions of the rays, without taking into account any lateral displacement.



Let SOs, in the second figure, be the direction of the incident ray, and take O the center of a spherical surface with radius unity; NON' the normal to the first surface, and OR the direction of the refracted ray; the plane of the first incidence being in the plane of the paper. Let $n'On$, in another plane, be the direction of the normal to the second surface of the prism, and OT, in the plane nOR, the direction of the finally emergent ray.

Then the letters of the figure being upon the surface of the sphere, and the dotted lines being supposed below the plane of the

paper; the arc Ts measures the angle of deviation $sOT=d$ which is to be found.

Let $\angle$ of incidence at first surface	$=\angle SON=\angle sON'$	$=i$
. . . refraction	$=\angle RON'$	$=i'$
. . . incidence at second surface	$=\angle ROn$	$=e'$
. . . emergence	$=TOn$	$=e$
. . . the angle of the prism	$=\angle N'On$	$=\alpha$
. . . the angle between the plane of first incidence and the perpendicular section of the prism	$=\angle RN'n$	$=\theta$

Then we shall have spherical triangles formed, which will be as in the lower figure, where the same letters refer to the same points, supposing the surface of the sphere near R to be seen more directly.

Now in triangles sRT, nRN' we have

$$\cos. sRT=\frac{\cos. Ts-\cos. Rs\,.\cos. RT}{\sin. Rs\,.\sin. RT}$$

$$=\cos. n\,RN'$$

$$=\frac{\cos. N'n-\cos. Rn\,.\cos. RN'}{\sin Rn\,.\sin. RN'}$$

or $$\frac{\cos. d-\cos.(i-i')\,.\cos.(e-e')}{\sin(i-i')\,.\sin.(e-e')}=\frac{\cos.\alpha-\cos. e'\,.\cos. i'}{\sin. e'\,.\sin. i'}$$

$$\therefore\ \cos. d=\frac{(\cos.\alpha-\cos. e'\cos. i')\sin.(i-i')\,.\sin.(e-e')}{\sin. e'\ \sin. i'}+\cos.(i-i')\,.\cos.(e-e')$$

we have also $\sin. i=\mu\sin. i'$, and $\sin. e=\mu\sin. e'$; also to connect the angles e' and i', we have from the triangle $N'Rn$

$$\cos.\theta=\cos. RN'n=\frac{\cos. Rn-\cos. RN'\,.\cos. N'n}{\sin. RN'\,.\sin. N'n}$$

whence $\quad \cos. e'=\cos.\theta\sin. i'\sin.\alpha+\cos. i'\cos.\alpha$
which gives e' when α, i' and θ are known.

If we make $\theta=0$ in the above expression, we have,

$$\cos. e'=\cos.(\alpha-i')\quad\text{or}\quad\alpha=i'+e'$$

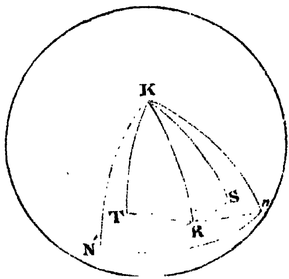
and substituting $\cos. \alpha = \cos. (i'+e')$ in the value of $\cos. d$, we find, after expanding $\cos. (i'+e')$,

$$\cos. d = \cos. (i-i'+e-e')$$

or $$d = i+e-i'-e' = i+e-\alpha$$ as in Art. 33, PART I.

ART. 28. PROP. *To shew that the incident ray and the emergent ray make equal angles with the edge of the prism.*

Let the points N', R, s, n, T be the same in the annexed figure as in the two latter ones of the last article. Take K a point on the surface of the sphere 90^0 distant from each of the points N' and n, where the normals meet it, then the line drawn from K to the center of the sphere is the direction of the edge of the prism.



Draw the arcs of great circles Ks, KR, KT, since Kn and KN' are each equal to 90^0, we have in the triangles $KN'R$, $KN's$,

$$\cos. KN'R = \frac{\cos. KR}{\sin. N'R}$$

$$= \frac{\cos. Ks}{\sin. N's}$$

$$\therefore \quad \cos. Ks = \cos. KR \frac{\sin. N's}{\sin. N'R} = \mu \cos. KR$$

and in triangles KnR, KnT we have

$$\cos. KnR = \frac{\cos. KR}{\sin. nR}$$

$$= \frac{\cos. KT}{\sin. nT}$$

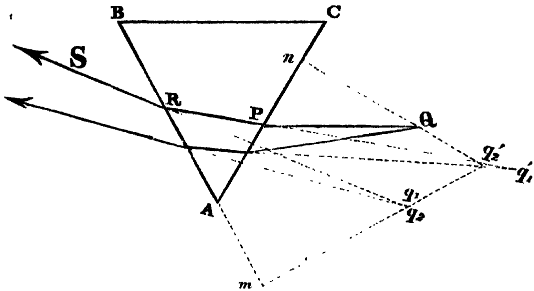
$$\therefore \quad \cos. KT = \cos. KR \frac{\sin. nT}{\sin. nR} = \mu \cos. KR$$

$\therefore$ $KT = Ks$, or the points where the directions of the incident and emergent rays meet the surface of the sphere are at equal distances from the point where the edge of the prism meets it;

that is, the incident and emergent rays make equal angles with the edge of the prism: also the angle which the refracted ray within the medium, measured by the arc KR, makes with the edge, is known from the expressions above found.

Art. 29. Prop. *To find the form of a small pencil of rays after traversing a prism, in a perpendicular section.*

Let ABC be the perpendicular section of the prism, Q the point from which the pencil diverges, $QPRS$ the path of the axis of



the pencil; draw nQ a perpendicular on the first surface of the prism, then by Article 23, q'_2 the place of the secondary focal line will be in it, and q'_1 the place of the primary focal line will be as in the figure. From q'_2 draw q'_2m a perpendicular on the second surface produced; then the positions of the primary and secondary foci of the emergent pencil will be at the points q_1 and q_2, of which q_2 is in q'_2m and q_1 very near it.

Let $QP=u$, $q'_1P=u'_1$, $q'_2P=u'_2$, $q_1R=v_1$, $q_2R=v_2$; let $\alpha=\angle BAC$ of the prism, i=angle of incidence of QP at P, i'=angle of refraction at P, e'=angle of incidence at R on the second surface, e=angle of emergence at R.

From Article 23 we have

$$u'_1=\mu u\frac{\cos.^2 i'}{\cos.^2 i}$$

$$u'_2=\mu u$$

Also, if the rays had traversed the prism in the contrary direction, converging to q_1 and q_2, the refracted rays would have converged to q'_1 and q'_2 respectively; therefore we have

$$q'_1R=\mu v_1 \frac{\cos.^2 e'}{\cos.^2 e}$$

$$q_2R=\mu v_2$$

Now,
$$q'_1R=u'_1+PR=u'_1+AP\,.\,\frac{\sin.\ RAP}{\sin.\ ARP}$$

Let AP, the distance of the point of incidence of the axis of the pencil, from A, $=a$; also $\angle ARP=180^0-(\alpha+APR)$ $=180^0-(\alpha+90^0-i')$

then
$$q'_1R=u'_1+a\,\frac{\sin.\ \alpha}{\cos.\,(\alpha-i')}$$

and
$$q'_2R=u'_2+PR$$

$$=u'_2+a\,\frac{\sin.\ \alpha}{\cos.\,(\alpha-i')}$$

Substituting the values of u'_1 and u'_2 we have,

$$\mu v_1\,\frac{\cos.^2 e'}{\cos.^2 e}=\mu u\,\frac{\cos.^2 i'}{\cos.^2 i}+a\,\frac{\sin.\ \alpha}{\cos.\,(\alpha-i')}$$

$$\mu v_2=\mu u+a\,\frac{\sin.\ \alpha}{\cos.\,(\alpha-i')}$$

whence

$$v_1=u\,\frac{\cos.^2 i'}{\cos.^2 i}\cdot\frac{\cos.^2 e}{\cos.^2 e'}+\frac{a\sin.\ \alpha}{\mu\cos.\ (\alpha-i')}\cdot\frac{\cos.^2 e}{\cos.^2 e'}$$

$$v_2=u+\frac{a\ \sin.\ \alpha}{\mu\ \cos.\ (\alpha-i')}$$

we see that in general there will be a circle of confusion; but at the angle of minimum deviation, when $i=e$, $i'=e'=\frac{\alpha}{2}$, we have

$$v_1=u+\frac{a\ \sin.\ \alpha}{\mu\ \cos.\ (\alpha-i')}\cdot\frac{\cos.^2 i}{\cos.^2 i'}$$

$$=u+\frac{2a\sin.\frac{\alpha}{2}}{\mu}\cdot\frac{1-\mu^2\sin.^2\frac{\alpha}{2}}{\cos.^2\frac{\alpha}{2}}$$

$$v_2=u+\frac{a\sin.\alpha}{\mu\cos.(\alpha-i')}$$

$$=u+\frac{2a\sin.\frac{\alpha}{2}}{\quad}$$

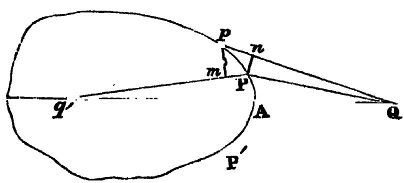
We see that near the angle of the prism the primary and secondary foci coincide very nearly; but that they separate more and more as the thickness, of the prism which is traversed, becomes greater.

The above discussion shews why the prism was directed to be set at the angle for the minimum deviation in the Chapter on Chromatics, PART I. pp. 107—8. It also points out the precautions to be taken when an exceedingly pure spectrum is required.

When the pencil is large, it must be remembered that, in accordance with Art. 22, it will be subject to aberration.

ART. 30. PROP. *To investigate the equation to the surface of accurate refraction.*

Let the pencil first diverge from Q, and after refraction converge to some point q' in the normal to the surface, QAq'. Let $QA=a$, $q'A=b$.



Taking any two rays of the pencil QP, Qp indefinitely near each other, let $QP=\rho$, $q'P=\rho'$. Draw the perpendiculars Pn and pm on Qp and $q'P$ respectively;

then $\qquad pn=$ increment of $\rho=d\rho$

$\qquad Pm=$ decrement of $\rho'=-d\rho'$

and if the arc $AP = s$, then $Pp = ds$; let i and i' be the angles of incidence and refraction, we have

$$Pm = Pp \,.\, \cos.\, pPm \quad \text{or} \quad -d\rho' = ds \sin.\, i'$$
$$pn = Pp \,.\, \cos.\, Ppn \quad \text{or} \quad d\rho = ds \sin.\, i$$
$$\therefore \quad d\rho = -\mu d\rho'$$

and integrating $\qquad \rho = -\mu\rho' + C$

for the direct ray QAq', we have

$$a + \mu b = C$$

$$\therefore \quad \rho = a + \mu b - \mu\rho' \ldots\ldots\ldots\ldots\ldots (1)$$

which is the property of the surface stated at the beginning of Chapter V, Part I.

To find the equation of the surface to polar co-ordinates, let $\angle PQA = \theta$, $\angle Pq'A = \theta'$. In the triangle QPq', we have the relations

$$\rho'^2 = \rho^2 + (a+b)^2 - 2\rho\,(a+b)\cos.\,\theta$$
$$\rho^2 = \rho'^2 + (a+b)^2 - 2\rho'(a+b)\cos.\,\theta'$$

between these equations and (1) we may eliminate ρ or ρ', and obtain the equation of the curve PAP', for q' or Q the pole respectively, as follows,

$$\rho'^2(\mu^2-1) - 2\rho'(\mu.\overline{a+\mu b} - \overline{a+b}\cos.\theta') + 2ab(\mu-1) + b^2(\mu^2-1) = 0 \quad (2)$$

$$\rho^2\left(\frac{\mu^2-1}{\mu^2}\right) + 2\rho\left(\frac{a+\mu b}{\mu^2} - \overline{a+b}\cos.\theta\right) + 2ab\left(\frac{\mu-1}{\mu}\right) + a^2\left(\frac{\mu^2-1}{\mu^2}\right) = 0 \quad (3)$$

If in (2) we take a indefinitely great, for the case of parallel incident rays, the terms into which it enters as a multiplier will be indefinitely great in comparison with the others which may therefore be neglected; also a being then a common factor, divides out, and we have

$$\rho' = \frac{b\,(\mu-1)}{\mu - \cos.\,\theta'} = \frac{b\left(1 - \dfrac{1}{\mu}\right)}{1 - \dfrac{1}{\mu}\cos.\,\theta'}$$

which is the equation of an ellipse referred to the further focus

(q') as pole, with eccentricity $e = \frac{1}{\mu}$, in accordance with Art. 37, PART I.

If in (3) we take b indefinitely great, for the case of the refracted rays being parallel, we need retain only those terms into which it enters as a multiplier, and afterwards omitting it as a common factor, we have

$$\rho = \frac{-a\left(\frac{\mu-1}{\mu}\right)}{\frac{1}{\mu} - \cos.\,\theta} = \frac{-a(\mu-1)}{1-\mu\cos.\,\theta}$$

which is the equation of an hyperbola referred to the further focus (Q) as pole, with eccentricity $e = \mu$, in accordance with Art. 38, PART I.

If a convergent incident pencil is to be refracted accurately to a focus, or a divergent incident pencil is to be refracted diverging accurately from a virtual focus, we find for both cases, using the same notation as before,

$$d\rho = \mu d\rho'$$

and

$$\rho = a - \mu b + \mu\rho'$$

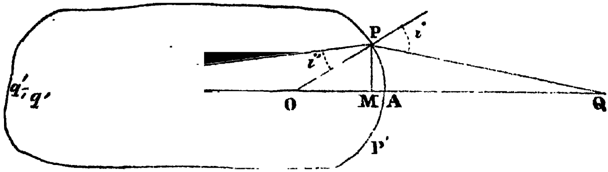
from which the polar equations of the curve, whose rotation about the normal ray QAq' generates the refracting surface, may be found as before.

From these the cases of accurate refraction at a spherical surface, as shewn at Art. 39, PART I. may be easily deduced.

ART. 31. PROP. *To find the form of the refracted pencil to a second approximation and the aberration of a given ray when a pencil is incident directly on a* convex *spherical refracting surface of a medium.*

Let Q be the point from which the incident pencil diverges; PAP' the section of the spherical refracting surface, of which the center is O; let $QAOq'$ be the ray which is incident perpendicularly at A; QP another ray refracted in Pq'.

If q_1 is the focus of the refracted rays, for an indefinitely small pencil, then $q'_1 q'$ is the aberration of the ray Pq'.



Draw PM perpendicular to QAO, and let $AM=x$, $PM=y$ be the co-ordinates of P, to the origin A; and if the radius $OA=OP=r$ we have $y^2=2rx-x^2$; let also $QA=u$, $q'A=u'$,

then
$$QP^2=QM^2+PM^2$$
$$=(u+x)^2+y^2$$
$$=(u+x)^2+2rx-x^2$$
$$=u^2+2x(u+r)$$
$$QP=u\sqrt{1+\frac{2x}{u^2}(u+r)}$$
$$=u\sqrt{1+2x\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)}$$

similarly
$$q'P^2=q'M^2+PM^2$$
$$=(u'-x)^2+y^2$$
$$=u'^2\left(1-\frac{2x}{u'^2}(u'-r)\right)$$

and
$$q'P=u'\sqrt{1-2x\frac{r}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)}$$

Now from the triangles QPO, $q'PO$ we have, as in Art. 43, Part I.

$$\mu\cdot\frac{q'O}{q'P}=\frac{QO}{QP}$$

or
$$\frac{\mu(u'-r)}{u'\sqrt{1-2x\frac{r}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)}}=\frac{u+r}{u\sqrt{1+2x\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)}}$$

dividing by r, &c.,

$$\frac{\mu}{r}-\frac{\mu}{u'}=\left(\frac{1}{r}+\frac{1}{u}\right)\left\{1+2x\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)\right\}^{-\frac{1}{2}}\times\left\{1-2x\frac{r}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)\right\}^{\frac{1}{2}}$$

Neglecting the terms involving x we obtain for u'_1 the first approximation

$$\mu\left(\frac{1}{r}-\frac{1}{u'}\right)=\frac{1}{r}+\frac{1}{u}$$

and

$$\frac{\mu}{u'_1}=\frac{\mu-1}{r}-\frac{1}{u}$$

as found in Art. 43, Part I.

If we substitute this value in the small term, we shall obtain a more correct value of u', by extracting the roots; and proceeding in the same manner we may find the value to any required accuracy, for any given value of x.

When x is small but not negligible, omitting powers of it above the first, we have,

$$\frac{\mu}{u'}=\frac{\mu}{r}-\left(\frac{1}{r}+\frac{1}{u}\right)\left\{1-x\left[\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)+\frac{r}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)\right]\right\}$$

$$=\frac{\mu-1}{r}-\frac{1}{u}+x\left\{\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)+\frac{r}{\mu^2}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left(\frac{1}{r}+\frac{1}{u}\right)\right\}\times\left(\frac{1}{r}+\frac{1}{u}\right)$$

$$=\frac{\mu-1}{r}-\frac{1}{u}+xr\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu^2-1}{\mu^2 u}+\frac{\mu-1}{\mu^2 r}\right)$$

$$=\frac{\mu-1}{r}-\frac{1}{u}+\frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)$$

If we substitute the approximate value $y^2=2rx$ nearly, we have the *second* approximation in the following form:

$$\frac{\mu}{u'}=\frac{\mu-1}{r}-\frac{1}{u}+\frac{y^2}{2}\cdot\frac{\mu-1}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)$$

and the aberration $=u'_1-u'=u'_1u'\left(\frac{1}{u'}-\frac{1}{u'_1}\right)$

$$=\frac{u'^2_1\cdot y^2(\mu-1)}{2\mu^3}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)\text{ nearly.}$$

When the second approximation only is wanted, we arrive at it more concisely as follows,

$$QP^2 = u^2 + 2x(u+r)$$

$$= u^2 + \frac{y^2}{r}(u+r) \text{ nearly}$$

$$\therefore \quad QP = u\left[1 + \frac{y^2}{2u}\left(\frac{1}{r}+\frac{1}{u}\right)\right] \text{ nearly}$$

and $$q'P^2 = u'^2 - 2x(u'-r)$$

$$= u'^2 - \frac{y^2}{r}(u'-r) \text{ nearly}$$

$$\therefore \quad q'P = u'\left[1 - \frac{y^2}{2u'}\left(\frac{1}{r}-\frac{1}{u'}\right)\right] \text{ nearly.}$$

substituting in $\mu \cdot \frac{q'O}{q'P} = \frac{QO}{QP}$,

we have $$\frac{\mu(u'-r)}{u'\left[1 - \frac{y^2}{2u'}\left(\frac{1}{r}-\frac{1}{u'}\right)\right]} = \frac{u+r}{u\left[1 + \frac{y^2}{2u}\left(\frac{1}{r}+\frac{1}{u}\right)\right]}$$

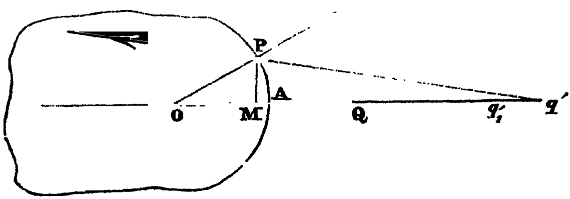
$$\therefore \mu\left(\frac{1}{r}-\frac{1}{u'}\right) = \left(\frac{1}{r}+\frac{1}{u}\right)\left\{1 - \frac{y^2}{2}\left[\frac{1}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)+\frac{1}{u}\left(\frac{1}{r}+\frac{1}{u}\right)\right]\right\}$$

$$= \left(\frac{1}{r}+\frac{1}{u}\right)\left\{1 - \frac{y^2}{2}\left[\frac{1}{\mu^2}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)+\frac{1}{u}\right]\left(\frac{1}{r}+\frac{1}{u}\right)\right\}$$

$$= \left(\frac{1}{r}+\frac{1}{u}\right)\left\{1 - \frac{y^2(\mu-1)}{2\mu^2}\left(\frac{\mu+1}{u}+\frac{1}{r}\right)\left(\frac{1}{r}+\frac{1}{u}\right)\right\}$$

$$\therefore \frac{\mu}{u'} = \frac{\mu-1}{r} - \frac{1}{u} + \frac{y^2(\mu-1)}{2\mu^2}\left(\frac{\mu+1}{u}+\frac{1}{r}\right)\left(\frac{1}{r}+\frac{1}{u}\right)^2$$

CASE 2. When Q is so near to A that q' is a virtual focus, as in the next figure, we have



$$q'P^2 = u'^2 + 2x(u'+r)$$

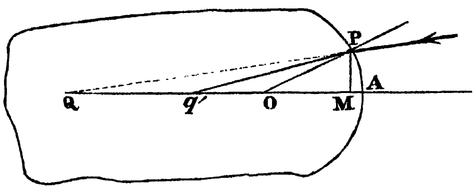
and $$q'P = u'\sqrt{1+\frac{2xr}{u}\left(\frac{1}{r}+\frac{1}{u'}\right)}$$

which gives

$$-\frac{\mu}{u'}=\frac{\mu-1}{r}-\frac{1}{u}+\frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)$$

and the aberration $q'q'_1$ has the same expression as before, but is now measured from A.

Case 3. When the incident pencil converges to Q, as in the next figure, we have



$$QP^2 = u^2 - 2x(u-r)$$

and $$QP = u\sqrt{1-\frac{2xr}{u}\left(\frac{1}{r}-\frac{1}{u}\right)}$$

which gives

$$\frac{\mu}{u'}=\frac{\mu-1}{r}+\frac{1}{u}-\frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}-\frac{1}{r}\right)$$

and the aberration is measured from the first approximate focus towards or from A according to the sign of the factor $\left(\frac{\mu+1}{u}-\frac{1}{r}\right)$; and vanishes when $(\mu+1)r=u$, which is the case of accurate refraction at a convex spherical surface investigated in Art. 39, Part I. The aberration also vanishes with the factor $\left(\frac{1}{r}-\frac{1}{u}\right)$ or when $u=r$, that is when Q coincides with O, and each ray enters the medium without deviation.

We see, that as in the cases of the first approximations, any one of the formulæ may be adapted to another case, by taking u and u' with contrary signs when measured in the opposite direction from A to that for the given formula.

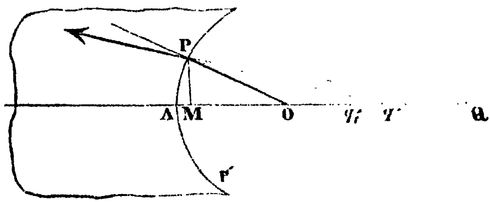
For a pencil of *parallel* incident rays, if we make $u=\infty$, we have

$$\frac{\mu}{u'}=\frac{\mu-1}{r}+\frac{x(\mu-1)}{\mu^2 r^2}$$
$$=\frac{\mu-1}{r}+\frac{y^2(\mu-1)}{2\mu^2 r^3}\ \text{nearly,}$$

which can be easily obtained by a direct solution.

ART. 32. PROP. *To find the form of the refracted pencil to a second approximation and the aberration of a given ray when a pencil is incident directly on a* concave *spherical refracting surface of a medium.*

In the figure, O being the center of the circular section PAP' of the spherical surface, let Q be the point from which the incident rays diverge; and QP being any incident ray, let $q'P$ be the direction of the refracted ray; $Qq'OA$ being the axis of the pencil.



Let $QA=u$, $q'A=u'$, $AO=r$: also, drawing PM perpendicular to AQ, let $AM=x$, $PM=y$; and $y^2=2rx-x^2$.

Now
$$QP^2=QM^2+PM^2$$
$$=(u-x)^2+y^2$$
$$=u^2-2x(u-r)$$

and
$$QP=u\sqrt{1-\frac{2xr}{u}\left(\frac{1}{r}-\frac{1}{u}\right)}$$

similarly
$$q'P^2=q'M^2+PM^2$$
$$=u'^2-2x(u'-r)$$

and
$$q'P=u'\sqrt{1-\frac{2xr}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)}$$

From the triangles QPO, $q'PO$, we have, as in Art. 44, PART I.,

$$\mu \cdot \frac{q'O}{q'P} = \frac{QO}{QP}$$

or
$$\frac{\mu(u'-r)}{u'\sqrt{1-\frac{2xr}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)}} = \frac{u-r}{u\sqrt{1-\frac{2xr}{u}\left(\frac{1}{r}-\frac{1}{u}\right)}}$$

whence

$$\frac{\mu}{u'} = \frac{\mu}{r} - \left(\frac{1}{r}-\frac{1}{u}\right)\left\{1-\frac{2xr}{u}\left(\frac{1}{r}-\frac{1}{u}\right)\right\}^{-\frac{1}{2}} \cdot \left\{1-\frac{2xr}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)\right\}^{\frac{1}{2}}$$

If we neglect the terms involving x, we have the first approximation

$$\frac{\mu}{u'_1} = \frac{\mu-1}{r} + \frac{1}{u}$$

and if we substitute this value of u' in the term with x, we shall have a more correct value of u'; and substituting the last found value again in the small term, we shall have a still more correct value of u', and so onwards until any required accuracy is attained.

When x is small, we obtain a more convenient expression by extracting the roots as far as the first power of x, and have

$$\frac{\mu}{u_1} = \frac{\mu-1}{r} + \frac{1}{u} - \left(\frac{1}{r}-\frac{1}{u}\right)\left\{\frac{xr}{u}\left(\frac{1}{r}-\frac{1}{u}\right) - \frac{xr}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)\right\}$$
$$= \frac{\mu-1}{r} + \frac{1}{u} - xr\left(\frac{1}{r}-\frac{1}{u}\right)^2 \cdot \left\{\frac{1}{u} - \frac{1}{\mu^2}\left(\frac{\mu-1}{r}+\frac{1}{u}\right)\right\}$$
$$= \frac{\mu-1}{r} + \frac{1}{u} - \frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}-\frac{1}{u}\right)^2 \cdot \left(\frac{\mu+1}{u}-\frac{1}{r}\right)$$

This expression might be deduced from that of the last article by putting $-r$ for r, and $-u'$ for u'.

For the aberration, taking u'_1 the first approximate value of u', we see that $\frac{\mu}{u'_1}$ is greater or less than $\frac{\mu}{u'}$, therefore u' is greater or

less than u'_1, or the first approximate focus q'_1 in the figure lies nearer to or further from A than q', according as the sign of the last factor in the small term is positive or negative.

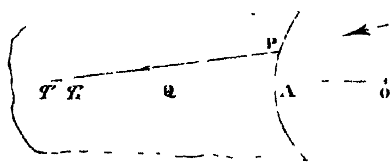
$$\text{The aberration} = u' - u'_1 = \frac{u'_1 . u'}{\mu}\left(\frac{\mu}{u'_1} - \frac{\mu}{u'}\right)$$

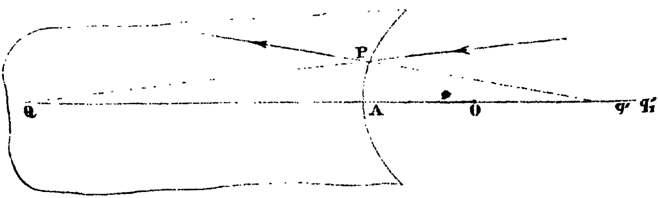
$$= \frac{u'^2_1 xr(\mu - 1)}{\mu^3}\left(\frac{1}{r} - \frac{1}{u}\right)^2 \cdot \left(\frac{\mu+1}{u} - \frac{1}{r}\right) \text{ nearly.}$$

We see that there are cases of no aberration for a diverging pencil incident on a concave refracting surface, as shewn in Art. 39, Part I., and the aberration changes sign as Q passes through the more distant of the aplanatic positions.

The other cases for pencils of converging rays, may be investigated in the same manner, or the results may be deduced from the previous ones by changing the signs of lines when they are to be measured in the contrary direction to that in the given case; as for instance,

Case 2. When the incident pencil converges to the point Q in the annexed figures, we find, by taking u negative in





the preceding expression,

$$\mp\frac{\mu}{u'} = \frac{\mu - 1}{r} - \frac{1}{u} + \frac{xr(\mu - 1)}{\mu^3}\left(\frac{1}{r} + \frac{1}{u}\right)^2\left(\frac{\mu+1}{u} + \frac{1}{r}\right)$$

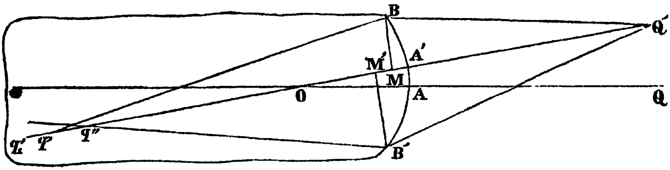
which applies to the upper figure when $\frac{1}{u} > \frac{\mu - 1}{r}$, and thence q'

falls on the same side of A with Q; and to the lower figure when $\frac{1}{u} < \frac{\mu - 1}{r}$ and q' falls on the opposite side of A from Q.

The aberration is measured in the first figure from A and in the second towards A, from the first approximate focus q'_1.

Art. 33. *To explain the change in the aberration when a pencil passes at a small obliquity through a circular aperture of a spherical refracting surface.*

Let BAB' be the section of the circular aperture by the plane of the paper; A being its center, and O the center of the curvature.



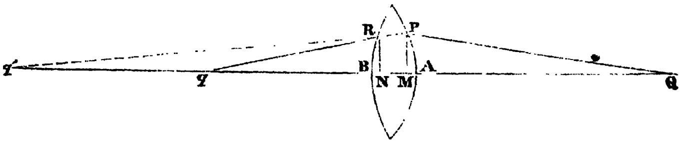
Then OAQ is the axis of the refracting surface; and if a pencil diverges from a point as Q', out of OQ, the line $Q'A'Oq'$ which enters the medium without deviation, may be considered the axis of the pencil. If q'_1 be the first approximate focus, q' the point where the ray refracted at the edge B of the aperture meets q'_1A'; and q'' the point where that refracted at the edge B' meets it; we shall have q'_1q'' greater than q'_1q', and the sections of the pencil near these points will not be symmetrical, but will produce figures analogous to the corresponding ones in mirrors as discussed in Art. 13.

Effects of a corresponding nature will arise in oblique refraction at concave surfaces.

Art. 34. Prop. *To find the form of the refracted pencil when a diverging pencil is incident directly on a double convex thin lens; and to find the aberration of a given ray.*

Let $QABq$ be the axis of the pencil incident directly on the lens; let QP be another ray of the incident pencil which is

refracted at the first surface in the direction PRq', and at the second surface in the direction Rq.



Let $AB=$ thickness of the lens $=t$; $QA=u$; $Aq'=u'$; $Bq=v$; let also $r=$ radius of the first surface, $s=$ that of the second; $x=AM$, $x'=BN$, $PM=y$.

From Art. 31, we have,

$$\frac{\mu}{u'}=\frac{\mu-1}{r}-\frac{1}{u}+\frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)$$

Considering the rays to pass in the contrary direction, diverging from q, we have from the second Case of Art. 31,

$$-\frac{\mu}{Bq'}=-\frac{\mu}{Aq'-t}=-\frac{\mu}{u'}-\frac{\mu t}{u'^2} \text{ nearly, since } t \text{ is small,}$$

$$=\frac{\mu-1}{s}-\frac{1}{v}+\frac{x's(\mu-1)}{\mu^2}\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)$$

Adding this latter expression to the former, and substituting the first approximate value of u' in the term containing t, we find,

$$\frac{1}{v}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2$$
$$+\frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+\frac{x's(\mu-1)}{\mu^2}\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)$$

Since PM and RN are very nearly equal in a thin lens, we have

$$y^2=2rx-x^2=2sx'-x'^2 \text{ very nearly,}$$

x and x' being very small, we have

$$rx=sx' \text{ very nearly;}$$

therefore, if we put $\frac{1}{f}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}$ we have,

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\frac{xr(\mu-1)}{\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)\right\}$$

If we put $xr=\frac{y^2}{2}$ nearly, we have the second approximation in the following form;

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\frac{y^2(\mu-1)}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)\right\}$$

The first approximate value of v being substituted in the small term of the above expressions, a more correct value will be obtained; and by successive substitutions, still more correct values.

Again, to find the value of the aberration, let v_1 be the first approximate value of v, or

$$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2$$

we have the longitudinal aberration

$$\begin{aligned}&=v_1-v\\&=v_1v\left(\frac{1}{v}-\frac{1}{v_1}\right)\\&=v_1^2\left(\frac{1}{v}-\frac{1}{v_1}\right)\text{ nearly.}\end{aligned}$$

Or, from the above expressions we have,

$$v_1-v=\frac{v_1^2xr(\mu-1)}{\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+\left(\frac{1}{s}+\frac{1}{v_1}\right)^2\left(\frac{\mu+1}{v_1}+\frac{1}{s}\right)\right\}$$

The values of this expression will change with u, r, and s, and values of r and s can be found, which will give a minimum aberration for given values of $xr\left(=\frac{y^2}{2}\text{ nearly}\right)$, μ, f and u; as will be found in the next article.

Art. 35. Prop. *To find the form of the lens for which the aberration of a given direct pencil is* a minimum; *and to find the positions of the foci when the aberration of a given lens is a* minimum.

In the last article we found the general expression for the aberration to be

$$\frac{v_1^2 xr(\mu-1)}{\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+\left(\frac{1}{s}+\frac{1}{v_1}\right)^2\left(\frac{\mu+1}{v_1}+\frac{1}{s}\right)\right\}$$

which, putting A for the expression between the large brackets, is

$$\frac{v_1^2 . xr(\mu-1)}{\mu^2} . A$$

The questions of minima depend on Av_1^2 for a ray of a pencil falling on the lens at a given distance y from the axis, and the expression for A involves the reciprocals of the four quantities r, s, u, v, to three dimensions; but when we substitute the values of s and v_1 in terms of r, f, and u, the second term contains the quantities of the first term, but affected with the negative sign, so that the highest powers go out and leave an equation of two dimensions only, in $\frac{1}{r}$ and $\frac{1}{u}$.

For substitution we have,

$$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u}$$

$$\frac{1}{s}=\frac{1}{(\mu-1)f}-\frac{1}{r}$$

which give

$$A=\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+\left\{\frac{\mu}{(\mu-1)f}-\left(\frac{1}{r}+\frac{1}{u}\right)\right\}^2\left\{\frac{\mu}{(\mu-1)f}-\left(\frac{\mu+1}{u}+\frac{1}{r}\right)\right\}$$

$$=\left\{\frac{\mu^2}{(\mu-1)^2f^2}-\frac{2\mu}{(\mu-1)f}\left(\frac{1}{r}+\frac{1}{u}\right)\right\}\left\{\frac{\mu}{(\mu-1)f}-\left(\frac{\mu+1}{u}+\frac{1}{r}\right)\right\}+\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu^2}{(\mu-1)f}\right)$$

$$=\frac{\mu(3\mu+2)}{(\mu-1)fu^2}+\frac{\mu(\mu+2)}{(\mu-1)fr^2}+\frac{4\mu(\mu+1)}{(\mu-1)fru}-\frac{\mu^2(3\mu+1)}{(\mu-1)^2f^2u}-\frac{\mu^2(2\mu+1)}{(\mu-1)^2f^2r}+\frac{\mu^4}{(\mu-1)^3f^3}$$

This expression is homogeneous in u, r, and f, and if we put $mu=f$, $nr=f$; after multiplying up the factor $\frac{\mu-1}{\mu}f^3$ we have

$$A\frac{\mu-1}{\mu}f^3=(3\mu+2)m^2+(\mu+2)n^2+4(\mu+1)mn-\frac{\mu(3\mu+1)}{\mu-1}m-\frac{\mu(2\mu+1)}{\mu-1}n+\frac{\mu^3}{(\mu-1)^2}$$

To find the form of the lens which has the least possible aber-

ration for given values of f and u, we must equate to zero the differential coefficient $\frac{dA}{dn}$, or we have

$$0=2(\mu+2)n+4(\mu+1)m-\frac{\mu(2\mu+1)}{\mu-1}$$

whence $$n=\frac{\mu(2\mu+1)}{2(\mu-1)(\mu+2)}-\frac{2(\mu+1)}{\mu+2}m$$

and n being calculated for given values of μ and m, we have r and s as required.

To find the positions of the foci when the aberration is a minimum or maximum negative for a given lens, we must have

$$\frac{d(Av_1^2)}{dv_1}=0$$

or $$2Av_1+v_1^2\frac{dA}{dv_1}=0$$

but $$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u}=\frac{1-m}{f}$$

and $$\frac{dv_1}{dm}=\frac{f}{(1-m)^2}$$

$$\therefore\quad 2A+(1-m)\frac{dA}{dm}=0$$

which gives a simple equation in m as follows,

$$0=2(\mu+2)n^2+4(\mu+1)nm-2\left(\frac{\mu+2}{\mu-1}\right)n+\frac{3\mu(\mu-1)-4}{\mu-1}m-\frac{\mu(\mu^2-2\mu-1)}{(\mu-1)^2}$$

or $$m=-\frac{2n[n(\mu-1)-1](\mu-1)(\mu+2)+\mu(1+2\mu-\mu^2)}{(\mu-1)[4n(\mu^2-1)+3\mu(\mu-1)-4]}$$

from which we can calculate the value of m when n and μ are given; and then also the magnitude of this minimum aberration.

Ex. 1. Required the form of the lens of least aberration for parallel incident rays, when $\mu=\frac{3}{2}$.

Taking the formula $n=\frac{\mu(2\mu+1)}{2(\mu+2)(\mu-1)}-\frac{2(\mu+1)}{\mu+2}m$

we have, $m=0$ since the incident rays are parallel,

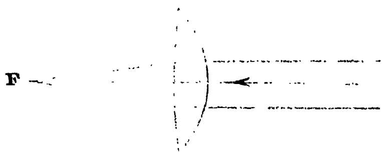
and $$n=\frac{12}{7}$$

$$\therefore \quad r=\frac{7}{12}f$$

and $$s=\frac{7}{2}f$$

or $$s=6r$$

This is the form of the lens long known to the practical opticians under the name of the *crossed* lens, as in the figure.



Ex. 2. Required the best form of a lens when the emergent rays are parallel, or when $u=f$.

We have now $m=-1$ since $u=f$

and $$n=\frac{12}{7}-\frac{10}{7}=\frac{2}{7}$$

$$\therefore \quad r=\frac{7}{2}f \text{ and } s=\frac{7}{12}f$$

or, the lens is the same in form as in the previous example, but with the less curved side to the incident rays.

Ex. 3. To find the value of the refractive index when the radius of the first surface taken equal to the focal length gives the best form of a lens, for parallel incident rays.

We have now $m=0$, and $n=1$.
Substituting these in the equation

$$0=2(\mu+2)n+4(\mu+1)m-\frac{\mu(2\mu+1)}{\mu-1}$$

we find $\mu=4$, but there is no substance known with this refractive power.

Ex. 4. To find the value of the refractive index when the plano-convex is the best form of a lens, for parallel incident rays.

We have now $m=0$ and $n=0$

and the above equation gives

$$\mu=-\frac{1}{2}$$

which is impossible, since the least value of μ must exceed unity when rays from a vacuum enter a dense medium.

Ex. 5. To find the refractive index when the convexo-plane is the best form of a lens, for parallel incident rays.

We have $m=0$ and $\dfrac{1}{s}=\dfrac{1-n(\mu-1)}{(\mu-1)f}=0$ since $s=\infty$

$$\therefore \quad n=\frac{1}{\mu-1}$$

and these substituted in the above equation give

$$\mu=1.686 \quad \text{and} \quad \mu=-1.186$$

The first is the refractive index of some of the denser glasses, and the latter is impossible.

Ex. 6. To find the positions of the conjugate foci when the longitudinal aberration of an equi-convex lens is a minimum.

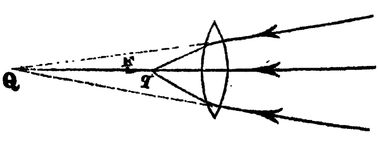
With $\mu=\dfrac{3}{2}$, we have now $n=1$

and $$m=-\frac{2n[n(\mu-1)-1](\mu-1)(\mu+2)+\mu(1+2\mu-\mu^2)}{(\mu-1)[4n(\mu^2-1)+3\mu(\mu-1)-4]}$$

$$=-\frac{7}{13}$$

and $u=-\dfrac{13}{7}f$, $v=\dfrac{13}{20}f$

or the incident pencil converges to Q, as in the figure.



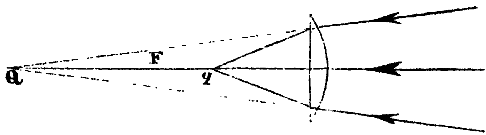
Ex. 7. To find the positions of the conjugate foci when the longitudinal aberration of the crossed-lens is a minimum.

We have now $n=\frac{12}{7}$, which gives $m=-\frac{99}{191}$

and

$$u=-\frac{191}{99}f,\ v=\frac{191}{290}f$$

or the incident pencil converges as in the figure.

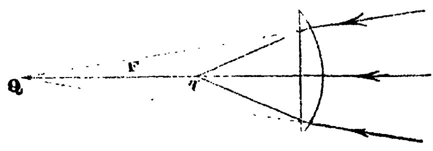


Ex. 8. To find the positions of the conjugate foci when the longitudinal aberration of the convexo-plane lens is a minimum.

We have now $n=2$, which gives $m=-\frac{7}{11}$

$$\therefore \quad u=-\frac{11}{7}f \text{ and } v=\frac{11}{18}f$$

and the conjugate foci are situated as in the figure.

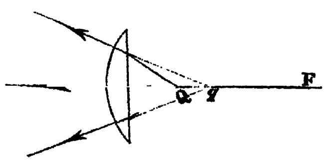


Ex. 9. To find the positions of the conjugate foci when the longitudinal aberration of a plano-convex lens is a minimum.

We have now $n=0$, which gives $m=3$

$$\therefore \quad u=\frac{1}{3}f \text{ and } v=-\frac{1}{2}f$$

and the incident pencil is now strongly divergent, as in the figure.



The following is a table of the coefficients in the aberrations of the different forms of lenses for *parallel* incident rays, when $\mu = \frac{3}{2}$.

Species of Lens.	Values of r and s in terms of f.	Value of A.	Coefficient of xr or $x's$.
Plano-convex . . .	$r = \infty,\quad s = \frac{1}{2}f$	$\frac{81}{2f^3}$	$\frac{9}{f}$
Convexo-plane . . .	$r = \frac{1}{2}f,\quad s = \infty$	$\frac{21}{2f^3}$	$\frac{7}{3f}$
Equi-convex	$r = f,\quad s = f$	$\frac{15}{f^3}$	$\frac{10}{3f}$
Crossed-lens	$r = \frac{7}{12}f,\quad s = \frac{7}{2}f$	$\frac{135}{14f^3}$	$\frac{15}{7f}$
Crossed-lens inverted .	$r = \frac{7}{2}f,\quad s = \frac{7}{12}f$	$\frac{435}{14f^3}$	$\frac{145}{21f}$

When rays fall on the different forms of lenses at equal distances from the axes, we have the ratios of the aberrations by comparing the values of A; thus the aberration in the plano-convex is nearly four times that of the convexo-plane lens.

If we take the second approximation by putting $\frac{y^2}{2}$ for xr, we find, as in Art. 9, the radius of the least circle of aberration

$$= \frac{y}{4v_1}(v_1 - v)$$

$$= \frac{y^3 v_1(\mu - 1)}{8\mu^2} A$$

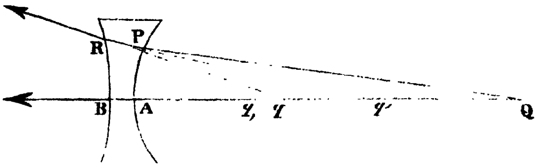
and its distance from the first approximate focus

$$= \frac{3}{4}(v_1 - v)$$

$$= \frac{3}{8} \cdot \frac{v_1^2 y^2(\mu - 1)}{\mu^2} A$$

Art. 36. Prop. *To find the form of the refracted pencil when a diverging pencil is incident directly on a double concave thin lens; and to find the aberration of a given ray.*

Let $QqAB$ be the axis of the lens, and the direction of the ray which falling perpendicularly on both surfaces, emerges without



deviation. Let Q be the focus of the incident rays; q' that of the refracted rays within the lens; and q the virtual focus of the emergent rays.

Let $AB=$ thickness of the lens, which is small, $=t$; let $QA=u$, $q'A=u'$, $qB=v$; r and s the radii of the first and second surfaces respectively, also x x' and y as in Art. 34.

By Art. 32 we have

$$\frac{\mu}{u'}=\frac{\mu-1}{r}+\frac{1}{u}-\frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}-\frac{1}{r}\right)$$

Considering the pencil to fall upon the lens in the contrary direction converging to q, we have from the second case of Art. 32,

$$-\frac{\mu}{Bq'}=\frac{\mu-1}{s}-\frac{1}{v}+\frac{x's(\mu-1)}{\mu^2}\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)$$

$$=-\frac{\mu}{u'+t}$$

$$=-\frac{\mu}{u'}+\frac{\mu t}{u'^2}$$

Adding to the former equation and substituting the first approximate value of u' in the term with t, we find,

$$\frac{1}{v}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}+\frac{1}{u}-\frac{t}{\mu}\left(\frac{\mu-1}{r}+\frac{1}{u}\right)^2-$$

$$-\frac{\mu-1}{\mu^2}\left\{xr\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}-\frac{1}{r}\right)-x's\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)\right\}$$

Also the distances of P and R from the axis being very nearly the same, we have, $\quad y^2=2rx-x^2=2sx'-x'^2$

$$\therefore \quad sx'=rx \text{ nearly}$$

and putting f for the principal focal length, we have

$$\frac{1}{v}=\frac{1}{f}+\frac{1}{u}-\frac{t}{\mu}\left(\frac{\mu-1}{r}+\frac{1}{u}\right)^2-$$

$$-\frac{xr(\mu-1)}{\mu^2}\left\{\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}-\frac{1}{r}\right)-\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)\right\}$$

putting $y^2=2rx$ we have the second approximation

$$\frac{1}{v}=\frac{1}{f}+\frac{1}{u}-\frac{t}{\mu}\left(\frac{\mu-1}{r}+\frac{1}{u}\right)^2+$$

$$+\frac{y^2(\mu-1)}{2\mu^2}\left\{\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)+\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{v}\right)\right\}$$

which we should have deduced from the corresponding expression in Art. 34, by changing the signs of r, s and v.

Putting v_1 for the first approximate value of v, we have, the aberration $=v_1-v=v_1v\left(\frac{1}{v}-\frac{1}{v_1}\right)$

$$=\frac{v_1^2y^2(\mu-1)}{2\mu^2}\left\{\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)+\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{v}\right)\right\}$$

which is to be measured towards A nearer than the first approximate focus q_1, since $\frac{1}{v_1}$ is less than $\frac{1}{v}$, or v_1 greater than v.

The other cases of convex and concave lenses may be investigated directly in the same way as the above, or the results may be obtained from any given formula taken as the standard case, by changing the signs of the lines which fall in a contrary direction to what they do in that case.

ART. 37. PROP. *To investigate the forms of the lenses for which the aberration vanishes at certain positions of the conjugate foci.*

In order that the aberration may vanish, we must have the quantity A of Art. 35 equal to zero, or

$$0=(3\mu+2)m^2+(\mu+2)n^2+4(\mu+1)mn-\frac{\mu(3\mu+1)}{\mu-1}m-\frac{\mu(2\mu+1)}{\mu-1}n+\frac{\mu^3}{(\mu-1)^2}$$

which being solved, gives us the two values of m in the following expression,

$$m=\frac{\mu(3\mu+1)-4(\mu^2-1)n\pm\sqrt{4\mu^2(\mu-1)^2n^2-4\mu^2(\mu-1)n-\mu^2(3\mu^2+2\mu-1)}}{2(\mu-1)(3\mu+2)}$$

but in order that m may be real, we must have the quantity under the radical sign equal to or greater than zero, or

$$4(\mu-1)^2n^2-4(\mu-1)n-(3\mu^2+2\mu-1)=\text{or}>0$$

or $$n=\text{or}>\frac{1\pm\sqrt{\mu(3\mu+2)}}{2(\mu-1)}$$

If we take $\mu=\frac{3}{2}$ this expression gives us,

$$n=\text{or}>1\pm3.12,$$

or $$n=\text{or}>4.12 \text{ and } -2.12$$

Substituting $n=4.12$ in the above expression for m we have

$$m=-1.9$$

and $$u=\frac{f}{m}=-\frac{f}{1.9}$$

Or, Q falls on the opposite side of the lens to that in the figure of Art. 34, and the incident pencil is strongly *convergent*.

Also $$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$

$$=\frac{2.9}{f}$$

$$\therefore \quad v=\frac{f}{2.9}$$

For the form of the lens we have

$$r=\frac{f}{n}=\frac{f}{4.12}$$

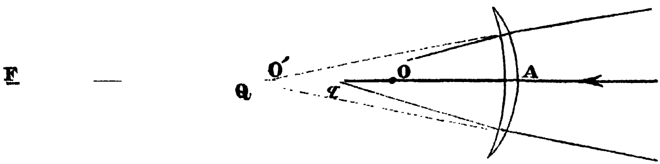
$$\frac{1}{s}=\frac{1}{(\mu-1)f}-\frac{1}{r}$$

$$=\frac{2}{f}-\frac{4.12}{f}$$

$$=-\frac{2.12}{f}$$

$$\therefore \quad s=-\frac{f}{2.12}$$

These data give a meniscus lens and Q, q the conjugate foci, as in the figure, drawn for a focal length of three inches.



Again substituting $n=-2.12$ in the expression for m we have

$$m=2.9$$

and

$$u=\frac{f}{m}=\frac{f}{2.9}$$

$$v=-\frac{f}{1.9}$$

$$r=\frac{f}{n}=-\frac{f}{2.12}$$

$$s=\frac{f}{4.12}$$

This is the same lens as before, with the concave surface towards the incident light, which diverges from q, and is refracted diverging accurately from the virtual focus Q.

The values of m will be real for all values of n positive or negative respectively, greater than the above; but for such lenses m will have two values which give $A=0$, and between these the

aberration changes sign. It is useful to examine a case amongst the forms of lenses which have this property of two *aplanatic foci*, *with the* aberration changing sign as the geometrical focus of the refracted pencil passes from one side to the other of these points.

Let $n=6$, we have,

$$r=\frac{f}{6}, \quad s=-\frac{f}{4}$$

also for $A=0$ we find

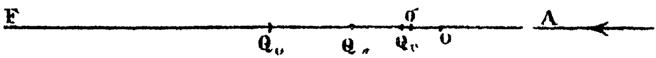
$$m=-2.44 \text{ and } m=-4.25$$

And again, if we put this value of n in the expression of Art. 35 for the aberration a minimum, (which is here a negative maximum) we find,

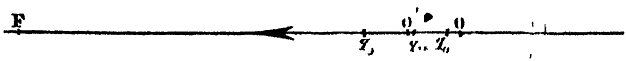
$$m=-3.16$$

which lies between the above.

The figures shew the form of the lens when of 3 inches focus,



and the positions of the conjugate foci, which for the sake of distinctness are marked on different figures; $Q_{,,}$ and $q_{,,}$ marking



the aplanatic foci, and Q_m q_m those at the negative maximum aberration.

We see that as n is larger the meniscus becomes deeper, and the aplanatic foci more separated. In all the cases of a positive focal length, whether n be positive or negative, the lens is a

meniscus, and the incident and emergent pencils have their conjugate aplanatic foci on the *concave* side of the lens.

Again, let f be negative, or the lens essentially concave, for *a single aplanatic focus we have as before $n=4.12$, and $n=-2.12$* which give,

$$r=\frac{-f}{4.12}, \quad s=\frac{-f}{2-4.12}=\frac{f}{2.12}$$

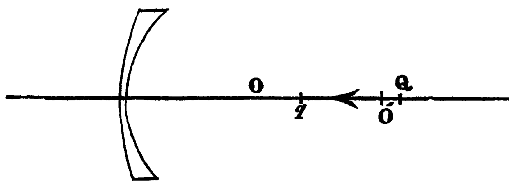
and $$r=\frac{-f}{-2.12}=\frac{f}{2.12}, \quad s=\frac{-f}{2+2.12}=\frac{-f}{4.12}$$

or the two values of n give the same *concavo-convex* lens, but turned the reverse ways to the incident light; the two corresponding values of m are $m=-1.9$ and $m=2.9$ as before

and $$u=\frac{-f}{m}=\frac{f}{1.9}, \quad \text{also } \frac{1}{v}=\frac{1}{f}-\frac{1}{u} \text{ generally.}$$

$$\therefore \quad v=\frac{-fu}{u+f}=-\frac{f}{2.9}$$

That is, in the first position of the lens, we have the incident pencil strongly divergent, as well as the refracted pencil, the positions of the lens and conjugate foci being as in the figure, which is drawn for a focal length of three inches.



In the second position, we have the convex surface turned towards the incident rays, which converging to q are refracted accurately converging to Q.

With greater positive and negative values of n, we have deeper *concavo-convex* lenses with each two aplanatic foci, and the aberration changes sign as the focus of the refracted pencil passes from one side to the other of those points.

These being the only cases in which the term which gives the

aberration can change sign, and as we have generally for convex lenses,

$$\frac{1}{v}=\frac{1}{v_1}+\frac{xr(\mu-1)}{\mu^2}A$$

therefore, when v_1 and v are positive

$$\frac{1}{v}>\frac{1}{v_1}$$

or $v<v_1$ and the aberration is measured *towards* the lens.

When v_1 and v are negative, or the image virtual

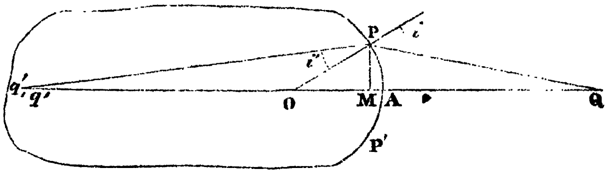
$$-\frac{1}{v}=-\frac{1}{v_1}+\frac{xr(\mu-1)}{\mu^2}A$$

and $$+\frac{1}{v}<\frac{1}{v_1}$$

or $v>v_1$ and the aberration is measured *from* the lens.

ART. 38. PROP. *To find the aberration of a spherical refracting surface when the pencil is so large that the second approximation is not sufficiently accurate.*

Referring to Art. 31, and using the same figure and letters, or



$$QA=u\,,\quad q'A=u'$$
$$OA=r\,,\quad AM=x\,,\quad PM=y$$

we have the fundamental equation

$$\frac{QO}{QP}=\mu\,.\,\frac{q'O}{q'P}\quad\ldots\ldots\ldots\ldots\quad(1)$$

and
$$QP = u\sqrt{1+2x\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)}$$

$$q'P = u'\sqrt{1-2x\frac{r}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)}$$

For convenience, put $z = 2x\,\frac{r}{u}\left(\frac{1}{r}+\frac{1}{u}\right)$, $z' = 2x\frac{r}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)$

we have

$$QP = u\sqrt{1+z}$$

$$q'P = u'\sqrt{1-z'}$$

substituting in (1)

$$\frac{\mu(u'-r)}{u'\sqrt{1-z'}} = \frac{u+r}{u\sqrt{1+z}}$$

or

$$\frac{\mu}{u'} = \frac{\mu}{r} - \left(\frac{1}{r}+\frac{1}{u}\right)(1+z)^{-\frac{1}{2}}(1-z')^{\frac{1}{2}}$$

$$= \frac{\mu}{r} - \left(\frac{1}{r}+\frac{1}{u}\right)\left(1-\frac{z}{2}+\frac{3z^2}{8}-\frac{5z^3}{16}+\&c.\right)\left(1-\frac{z'}{2}-\frac{z'^2}{8}-\frac{z'^3}{16}-\&c.\right)$$

$$= \frac{\mu}{r} - \left(\frac{1}{r}+\frac{1}{u}\right)\left(1-\frac{1}{2}(z+z')+\frac{zz'}{4}+\frac{1}{8}(3z^2-z'^2)-\&c.\right)$$

$$= \frac{\mu-1}{r} - \frac{1}{u} + \left(\frac{1}{r}+\frac{1}{u}\right)\left\{\frac{z+z'}{2}-\frac{zz'}{4}-\frac{3z^2-z'^2}{8}+\&c.\right\}$$

From this the second approximation is obtained as in Art. 31, by neglecting terms with powers and products of z and z' above the first:

or
$$\frac{\mu}{u'} = \frac{\mu-1}{r} - \frac{1}{u} + \frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)$$

For the *third* approximation, retaining all the terms with x^2, we have

$$z' = 2x\frac{r}{u'}\left(\frac{1}{r}-\frac{1}{u'}\right)$$

$$=\frac{2xr}{\mu^2}\Big[\frac{\mu-1}{r}-\frac{1}{u}+$$

$$+\frac{xr(\mu-1)}{\mu^2}\Big(\frac{1}{r}+\frac{1}{u}\Big)^2\Big(\frac{\mu+1}{u}+\frac{1}{r}\Big)\Big]\Big[1-\frac{xr(\mu-1)}{\mu^2}\Big(\frac{1}{r}+\frac{1}{u}\Big)\Big(\frac{\mu+1}{u}+\frac{1}{r}\Big)\Big]$$

$$\times\Big(\frac{1}{r}+\frac{1}{u}\Big)$$

$$=\frac{2xr}{\mu^2}\Big(\frac{1}{r}+\frac{1}{u}\Big)\Big(\frac{\mu-1}{r}-\frac{1}{u}\Big)+$$

$$+\frac{2x^2r^2(\mu-1)}{\mu^4}\Big(\frac{1}{r}+\frac{1}{u}\Big)^2\Big(\frac{\mu+1}{u}+\frac{1}{r}\Big)\Big(\frac{2-\mu}{r}+\frac{2}{u}\Big)$$

and $\quad z'^2=\dfrac{4x^2r^2}{\mu^4}\Big(\dfrac{1}{r}+\dfrac{1}{u}\Big)^2\Big(\dfrac{\mu-1}{r}-\dfrac{1}{u}\Big)^2$ as far as x^2

$$zz'=\frac{2xr}{u}\Big(\frac{1}{r}+\frac{1}{u}\Big)\times\frac{2xr}{\mu^2}\Big(\frac{1}{r}+\frac{1}{u}\Big)\Big(\frac{\mu-1}{r}-\frac{1}{u}\Big)\text{ as far as }x^2$$

$$=\frac{4x^2r^2}{\mu^2u}\Big(\frac{1}{r}+\frac{1}{u}\Big)^2\Big(\frac{\mu-1}{r}-\frac{1}{u}\Big)$$

Substituting these values we have

$$\frac{\mu}{u'}=\frac{\mu-1}{r}-\frac{1}{u}+\Big(\frac{1}{r}+\frac{1}{u}\Big)\Big\{\frac{z+z'}{2}-\frac{zz'}{4}-\frac{3z^2-z'^2}{8}\Big\}$$

$$=\frac{\mu-1}{r}-\frac{1}{u}+\frac{xr(\mu-1)}{\mu^2}\Big(\frac{1}{r}+\frac{1}{u}\Big)^2\Big(\frac{\mu+1}{u}+\frac{1}{r}\Big)+$$

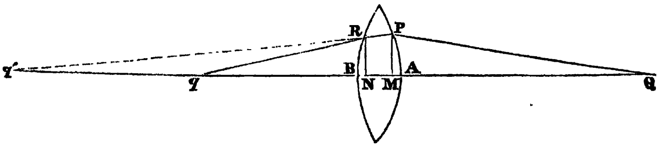
$$+x^2r^2\Big(\frac{1}{r}+\frac{1}{u}\Big)^3\Big\{\frac{\mu-1}{\mu^4}\Big(\frac{\mu+1}{u}+\frac{1}{r}\Big)\Big(\frac{2-\mu}{r}+\frac{2}{u}\Big)-\frac{3}{2u^2}+$$

$$+\frac{1}{2\mu^4}\Big(\frac{\mu-1}{r}-\frac{1}{u}\Big)\Big(\frac{\mu-1}{r}-\frac{2\mu^2+1}{u}\Big)\Big\}$$

If we put u'_1 the first approximate value of u', we have the aberration $=u'_1-u'=u_1'u'\left(\dfrac{1}{u'}-\dfrac{1}{u'_1}\right)$, which must contain all the terms with x^2.

ART. 39. PROP. *A diverging pencil of rays falling on a double convex lens, required the form of the emergent pencil to a third approximation, and the aberration.*

Using the figure and notation of Art. 34, where $QABqq'$ is the axis of the lens and the direction of the ray which has no



deviation; let QP be another ray refracted in PRq' within the lens, and finally emergent in the direction Rq.

By the last proposition we have

$$\frac{\mu}{u'}=\frac{\mu-1}{r}-\frac{1}{u}+xr\frac{(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)$$

$$+x^2r^2\left(\frac{1}{r}+\frac{1}{u}\right)^3\left\{\frac{\mu-1}{\mu^4}\left(\frac{\mu+1}{u}+\frac{1}{r}\right)\left(\frac{2-\mu}{r}+\frac{2}{u}\right)-\frac{3}{2u^2}+\right.$$

$$\left.+\frac{1}{2\mu^4}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left(\frac{\mu-1}{r}-\frac{2\mu^2+1}{u}\right)\right\}$$

or, putting B for the part of the last term within the large brackets,

$$\frac{\mu}{u'}=\frac{\mu-1}{r}-\frac{1}{u}+xr\frac{(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+x^2r^2\left(\frac{1}{r}+\frac{1}{u}\right)^3B.$$

Now, supposing the lens very thin, and the rays qB, qR to fall upon the lens, diverging from q, and to be refracted diverging from q', we have

$$-\frac{\mu}{u'}=\frac{\mu-1}{s}-\frac{1}{v}+x's\frac{(\mu-1)}{\mu^2}\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)+x'^2s^2\left(\frac{1}{s}+\frac{1}{v}\right)^3B',$$

where B' is formed by putting s for r and v for u in the expression for B.

Adding to the former we have

$$\frac{1}{v}=(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)-\frac{1}{u}+xr\frac{(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+$$

$$+x's\frac{(\mu-1)}{\mu^2}\left(\frac{1}{s}+\frac{1}{v}\right)^2\left(\frac{\mu+1}{v}+\frac{1}{s}\right)$$

$$+x^2r^2\left(\frac{1}{r}+\frac{1}{u}\right)B+x'^2s^2\left(\frac{1}{s}+\frac{1}{v}\right)B$$

In this expression we must use the first approximate value of v, say v_1, in the term with x'^2, but we must use the second approximate value in the term with x', since we retain all terms of the order x^2; also the lens being thin, we have the ordinates RN and PM equal, very nearly,

$$\therefore\quad y^2=2rx-x^2=2sx'-x'^2$$

or

$$x's=xr+\frac{x'^2-x^2}{2}$$

and $\quad x'=\frac{xr}{s}$ nearly, as used for the second approximation.

$$\therefore\quad x's=xr+\frac{x^2}{2}\left(\frac{r^2-s^2}{s^2}\right)=xr+\frac{x^2r^2}{2}\left(\frac{1}{s^2}-\frac{1}{r^2}\right)\text{ very nearly.}$$

This value of $x's$ we must use, since we retain terms with x^2; but we must use in the last term $x'^2s^2=x^2r^2$, for our *third approximation*. Now for the second approximation we have, as in Art. 34,

$$\frac{1}{v}=(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)-\frac{1}{u}+\frac{xr(\mu-1)}{\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+\right.$$

$$\left.+\left(\frac{1}{s}+\frac{1}{v_1}\right)^2\left(\frac{\mu+1}{v_1}+\frac{1}{s}\right)\right\}$$

or putting A for the quantity between the large brackets, as in Art. 35, we have,

$$\frac{1}{v}=(\mu-1)\left(\frac{1}{r}+\frac{1}{u}\right)-\frac{1}{u}+\frac{xr(\mu-1)}{\mu^2}A$$

$$=\frac{1}{v_1}+\frac{xr(\mu-1)}{\mu^2}A$$

Substituting this value, and the value of $x's$ above, in the former equation, we have for the third approximation,

$$\frac{1}{v}=(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)-\frac{1}{u}+\frac{xr(\mu-1)}{\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{\mu+1}{u}+\frac{1}{r}\right)+$$

$$+x^2r^2\left\{\left(\frac{1}{r}+\frac{1}{u}\right)^3B+\left(\frac{1}{s}+\frac{1}{v_1}\right)^3B'\right\}+\left[xr+\frac{x^2r^2}{2}\left(\frac{1}{s^2}-\frac{1}{r^2}\right)\right]\times$$

$$\times\frac{\mu-1}{\mu^2}\left(\frac{1}{s}+\frac{1}{v_1}+\frac{xr(\mu-1)}{\mu^2}A\right)^2\left(\frac{\mu+1}{v_1}+\frac{1}{s}+\frac{xr(\mu^2-1)}{\mu^2}A\right)$$

$$=\frac{1}{v_1}+\frac{xr(\mu-1)}{\mu^2}A+x^2r^2\left\{\left(\frac{1}{r}+\frac{1}{u}\right)^3B+\left(\frac{1}{s}+\frac{1}{v_1}\right)^3B'+\right.$$

$$+\frac{\mu-1}{2\mu^2}\left(\frac{1}{s^2}-\frac{1}{r^2}\right)\left(\frac{1}{s}+\frac{1}{v_1}\right)^2\left(\frac{\mu+1}{v_1}+\frac{1}{s}\right)+$$

$$\left.+\frac{(\mu-1)^2}{\mu^4}\left(\frac{1}{s}+\frac{1}{v_1}\right)\left(\frac{3(\mu+1)}{v_1}+\frac{\mu+3}{s}\right)A\right\}$$

and the aberration to a third approximation is

$$v_1-v=v_1v\left(\frac{1}{v}-\frac{1}{v_1}\right)$$

in which we must retain all the terms containing x^2.

Now $\frac{1}{v}=\frac{1}{v_1}+\frac{xr(\mu-1)}{\mu^2}A$, to the second approximation

$$=\frac{1}{v_1}\left(1+\frac{xr(\mu-1)}{\mu^2}Av_1\right)$$

and $\quad\therefore\quad vv_1=v_1^2\left(1-\frac{xr(\mu-1)}{\mu^2}Av_1\right)$

and the aberration $=v_1-v$, by putting C for the coefficient of x^2r^2,

$$=v_1^2\left(1-\frac{xr(\mu-1)}{\mu^2}Av_1\right)\left(\frac{xr(\mu-1)}{\mu^2}A+x^2r^2C\right)$$

$$=v_1^2\left[\frac{xr(\mu-1)}{\mu^2}A+x^2r^2\left(C-\frac{(\mu-1)^2}{\mu^4}A^2.v_1\right)\right]$$

Art. 40. Prop. *To transform the expression for $\frac{1}{r}$ into another with the quantities of Art. 35.*

As in Art. 35 we have $mu=f$, $nr=f$

$$\therefore \quad \frac{1}{u}=\frac{m}{f}, \quad \frac{1}{r}=\frac{n}{f}$$

$$\frac{1}{r_1}=\frac{1}{f}-\frac{1}{u}=\frac{1-m}{f}$$

$$\frac{1}{s}=\frac{1}{(\mu-1)f}-\frac{1}{r}=\frac{1}{f}\left(\frac{1}{\mu-1}-n\right)$$

These give, as in Art. 35.

$$A=\frac{\mu}{(\mu-1)f^3}\left\{(3\mu+2)m^2+(\mu+2)n^2+4(\mu+1)mn-\frac{\mu(3\mu+1)}{\mu-1}m-\frac{\mu(2\mu+1)}{\mu-1}n+\frac{\mu^2}{(\mu-1)^2}\right\}$$

and also for the quantities B and B' first used in the last article, the following values

$$B=\frac{\mu-1}{\mu^4}\left(\frac{\mu+1}{u}+\frac{1}{r}\right)\left(\frac{2-\mu}{r}+\frac{2}{u}\right)-\frac{3}{2u^2}+$$

$$+\frac{1}{2\mu^4}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left(\frac{\mu-1}{r}-\frac{2\mu^2+1}{u}\right)$$

$$=\frac{1}{2\mu^4 f^2}\left\{(\mu-1)(3-\mu)n^2+2(\mu^2-1)(3-2\mu)mn-3(\mu^2-1)^2m^2\right\}$$

$$B'=\frac{\mu-1}{\mu^4}\left(\frac{\mu+1}{v_1}+\frac{1}{s}\right)\left(\frac{2-\mu}{s}+\frac{2}{r_1}\right)-\frac{3}{2r_1^2}+$$

$$+\frac{1}{2\mu^4}\left(\frac{\mu-1}{s}-\frac{1}{v_1}\right)\left(\frac{\mu-1}{s}-\frac{2\mu^2+1}{v_1}\right)$$

$$=B+\frac{1}{\mu^3 f^2}\left\{n(2\mu^2-3\mu-1)+m(\mu+1)[3\mu(\mu-1)-1]+\frac{\mu^2[2-3\mu(\mu-1)]}{2(\mu-1)}\right\}$$

and substituting in the same manner in the other coefficients, we have

$$\frac{1}{v}=\frac{1}{v_1}+\frac{xr(\mu-1)}{\mu^2}A+\frac{x^2r^2}{f^5}\Big\{(m+n)^3B.f^2+\Big(\frac{\mu}{\mu-1}-(m+n)\Big)^3B'.f^2+$$
$$+\frac{(\mu-1)^2}{\mu^4}\Big(\frac{\mu}{\mu-1}-(m+n)\Big)\Big(\frac{\mu+3}{\mu-1}+3(\mu+1)(1-m)-(\mu+3)n\Big)A.f^3+$$
$$+\frac{1}{2\mu^2}\Big(\frac{1}{\mu-1}-2n\Big)\Big(\frac{\mu}{\mu-1}-(m+n)\Big)^2\Big(\frac{\mu^2}{\mu-1}-(\mu+1)m-n\Big)\Big\}$$

or, now substituting for B' its value, we have

$$\frac{1}{v}=\frac{1}{v_1}+\frac{xr(\mu-1)}{\mu^2}A+\frac{x^2r^2}{f^5}\Big\{\Big(\frac{\mu^3}{(\mu-1)^3}-\frac{3\mu^2(m+n)}{(\mu-1)^2}+\frac{3\mu(m+n)^2}{\mu-1}\Big)Bf^2+$$
$$+\Big(\frac{\mu}{\mu-1}-(m+n)\Big)^3\times$$
$$\times\frac{1}{\mu^3}\Big(n(2\mu^2-3\mu-1)+m(\mu+1)[3\mu(\mu-1)-1]+\frac{\mu^2[2-3\mu(\mu-1)]}{2(\mu-1)}\Big)$$
$$+\frac{(\mu-1)^2}{\mu^4}\Big(\frac{\mu}{\mu-1}-(m+n)\Big)\Big(\frac{\mu+3}{\mu-1}+3(\mu+1)(1-m)-(\mu+3)n\Big)Af^3$$
$$+\frac{1}{2\mu^2}\Big(\frac{1}{\mu-1}-2n\Big)\Big(\frac{\mu}{\mu-1}-(m+n)\Big)^2\Big(\frac{\mu^2}{\mu-1}-(\mu+1)m-n\Big)\Big\}$$

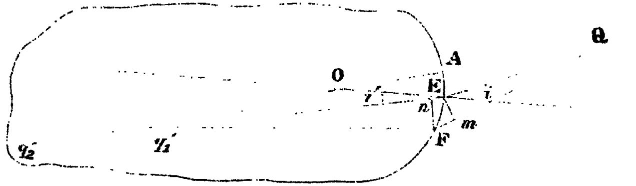
we see that m and n occur to the fourth power, when we have the value of $\frac{1}{v}$ to the third approximation; whilst to the second approximation, as in Art. 35, they rise only to the second power.

Art. 41. Prop. *To find the primary and secondary foci when a small pencil of diverging rays is incident* obliquely *on a convex spherical refracting surface.*

Let Q be the point from which the incident pencil diverges, QE the axis of the pencil, and QF another ray indefinitely near to it.

Let the refracted rays $Eq'_1q'_2$, Fq'_1 intersect in q'_1 the primary focus; and let the refracted ray $Eq'_1q'_2$ meet the line $QAOq'_2$,

passing through the center of curvature O, in q_2' the secondary focus.



Let r = radius of the surface $= OA$ or OE, $u = QE$, $u'_1 = q'_1E$, $u'_2 = q'_2E$. From E draw Em perpendicular to QF, and from F draw Fn perpendicular to q'_1E, then i and i' being the angles of incidence and refraction,

$$Fm = \text{increment of } u = du$$
$$En = \text{decrement of } u'_1 = du'_1$$

and ultimately
$$En = EF \cos. FEn = EF \sin. i'$$
$$Fm = EF \sin. FEm = EF \sin. i$$
$$\therefore \quad du = \mu du'_1$$

Joining q'_1 and O, we have

in the triangle q'_1OE, $\quad q'_1O^2 = q'_1E^2 + OE^2 - 2 . q'_1E . OE \cos. OEq'_1$
$$= u'^2_1 + r^2 - 2ru'_1 \cos. i'$$
and in triangle QOE, $\quad QO^2 = QE^2 + OE^2 - 2 . QE . OE \cos. QEO$
$$= u^2 + r^2 + 2ur \cos. i$$

Now QO and q'_1O are constant whilst u, u'_1, i and i' vary from the point E to the point F, therefore differentiating, we have

$$0 = u'_1 \, du'_1 - r \cos. i' . du'_1 + ru'_1 \sin. i' . di'$$
$$0 = u \, du + r \cos. i . du - ru \sin. i . di$$

$$\therefore \quad \sin. i' . di' = -\frac{u'_1 - r\cos. i'}{ru'_1} du'_1$$

$$\sin. i . di = \frac{u + r\cos. i}{ru} du .$$

But $\sin. i = \mu \sin. i'$, $di = \mu \frac{\cos. i'}{\cos. i}\, di'$

and $du = -\mu\, du'_1$, hence equating the values of $\sin. i \,.\, di$

we have, $$\mu^2 \frac{\cos. i'}{\cos. i} \sin. i'\, di' = \frac{u + r \cos. i}{ru}\, du$$

$$= \frac{u'_1 - r \cos. i'}{ru'_1} \cdot du \cdot \frac{\mu \cos. i'}{\cos\ i}$$

or $$\frac{1}{r} + \frac{\cos. i}{u} = \frac{\mu \,.\, \cos. i'}{\cos. i} \cdot \frac{1}{r} - \frac{\mu \cos. i'}{\cos. i} \cdot \frac{\cos. i'}{u'_1} \,.$$

and $$\frac{\mu \cos. i'}{\cos. i} \cdot \frac{\cos\ i'}{u'_1} = \left(\frac{\mu \cos. i'}{\cos. i} - 1\right)\frac{1}{r} - \frac{\cos. i}{u} \quad \ldots\ldots\ldots \quad (1$$

which gives u'_1 the primary focal distance.

Again, as in the previous propositions, we have

$$\frac{\mu \,.\, q'_2 O}{q'_2 E} = \frac{QO}{QE}\,, \qquad \text{or} \qquad \frac{\mu^2 \,.\, q'_2 O^2}{q'_2 E^2} = \frac{QO^2}{QE^2}$$

or $$\frac{\mu^2 (u'^2_2 + r^2 - 2u'_2 r \cos. i')}{u'^2_2} = \frac{u^2 + r^2 + 2ru \cos. i}{u^2}$$

dividing by r^2 &c.

$$\frac{\mu^2}{r^2} + \frac{\mu^2}{u'^2_2} - \frac{2\mu^2 \cos. i'}{u'_2 r} = \frac{1}{r^2} + \frac{1}{u^2} + \frac{2 \cos. i}{ur}$$

subtract $\frac{\mu^2 \sin.^2 i'}{r^2} = \frac{\sin.^2 i}{r^2}$ from the respective sides of the last equation, and we have

$$\frac{\mu^2 \cos.^2 i'}{r^2} - \frac{2\mu^2 \cos. i'}{ru'_2} + \frac{\mu^2}{u'^2_2} = \frac{\cos.^2 i}{r^2} + \frac{2 \cos. i}{ru} + \frac{1}{u^2}$$

which are exact squares, and extracting the roots

$$\frac{\mu \cos. i'}{r} - \frac{\mu}{u'_2} = \frac{\cos.\ i}{r} + \frac{1}{u}$$

$$\therefore \quad \frac{\mu}{u'_2} = \left(\frac{\mu \cos. i'}{\cos. i} - 1\right)\frac{\cos.\ i}{r} - \frac{1}{u} \quad \ldots\ldots\ldots \quad (2)$$

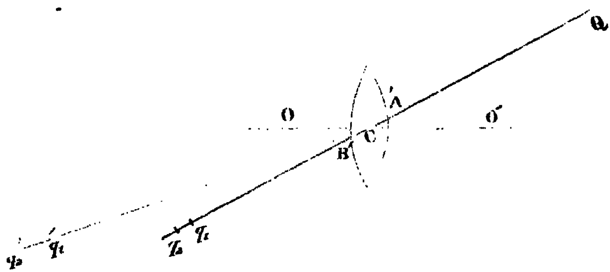
which gives u'_2 the secondary focal distance.

If we make i and $i'=0$, we have $u'_1=u'_2$, and the result is the first approximate value, the expressions we have investigated applying only to very small pencils, and the aberration being neglected.

The case of a concave surface and a small oblique pencil may be investigated independently, or deduced in the usual manner from the above, as well as the cases for converging incident pencils, and pencils diverging from points so near the lens that q'_1 and q'_2 fall on the same side of the lens as Q, when u_1 and u_2 become negative in the above equations.

ART. 42. PROP. *To find the primary and secondary foci, when a small diverging pencil passes obliquely but centrically through a thin double convex lens.*

Let Q be the point from which the incident pencil diverges, and $QA'CB'q_1q_2$, the course of the ray which is the axis of the pencil,



passing within the lens through the center C; so that O and O' being the centers of the curvatures of the surfaces, the radii OA' and $O'B'$ to the points of incidence and emergence are parallels, and the angle of incidence at A' equals the angle of emergence at $B'=i$ say; see Art. 71, PART I.

Let $QA'=u$, $q'_1A'=u'_1$, $q'_2A'=u'_2$, $q_2B'=v_2$, $q_1B'=v_1$ and $t=$ thickness of the lens, t sec. $i'=A'B'$ very nearly.

At the first refraction we have, by the last Art.

$$\frac{\mu \cos. i}{\cos. i} \cdot \frac{\cos. i'}{u'_1} = \left(\mu \frac{\cos. i'}{\cos. i} - 1\right)\frac{1}{r} - \frac{\cos. i}{u}$$

$$\frac{\mu}{u'_2} = \left(\mu \frac{\cos. i'}{\cos. i} - 1\right)\frac{\cos. i}{r} - \frac{1}{u}$$

At the second refraction, considering the pencil to pass in the contrary direction, we have

$$-\frac{1}{B'q'_1} = -\frac{1}{A'q'_1 - t \sec. i'} = -\frac{1}{u'_1} - \frac{t \sec. i'}{u'^2_1}$$

$$-\frac{1}{B'q'_2} = -\frac{1}{A'q'_2 - t \sec. i'} = -\frac{1}{u'_2} - \frac{t \sec. i'}{u'^2_2}$$

and

$$-\mu \frac{\cos. i'}{\cos. i} \cdot \frac{\cos. i'}{B'q'_1} = \left(\mu \frac{\cos. i'}{\cos. i} - 1\right)\frac{1}{s} - \frac{\cos. i}{v_1}$$

$$= -\mu \frac{\cos. i'}{\cos. i} \frac{\cos. i'}{u'_1} - \mu \frac{\cos. i'}{\cos. i} \cdot \frac{t}{u'^2_1}$$

$$-\frac{\mu}{B'q'_2} = \left(\mu \frac{\cos. i'}{\cos. i} - 1\right)\frac{\cos. i}{s} - \frac{1}{v_2}$$

$$= -\frac{\mu}{u'_2} - \frac{\mu\, t \sec. i'}{u'^2_2}$$

Adding to the former expressions, we have

$$\frac{1}{v_1} = \left(\mu \frac{\cos. i'}{\cos. i} - 1\right)\left\{\frac{1}{r} + \frac{1}{s}\right\} \sec. i - \frac{1}{u} + \mu \frac{\cos. i'}{\cos.^2 i} \cdot \frac{t}{u'^2_1}$$

$$\frac{1}{v_2} = \left(\mu \frac{\cos. i'}{\cos. i} - 1\right)\left\{\frac{1}{r} + \frac{1}{s}\right\} \cos. i - \frac{1}{u} + \mu \frac{t \sec. i'}{u'^2_2}$$

When the lens is very thin compared with the values of u'_1 and u'_2 we may neglect the term with t, but otherwise the positions of q_1 and q_2 are affected by the thickness.

The values of v_1 and v_2 being found from the above expressions, the magnitude of the circle of confusion and its distance from the lens can be calculated as in Art. 12, but the pencil must be considered small compared with the obliquity.

ART. 43. PROP. *To find the* approximate *expressions for the primary and secondary foci, when a small centrical diverging pencil passes at a* small obliquity *through a thin double convex lens.*

The expressions found in the last proposition can be brought to approximate forms, when the obliquity is *small,* which are frequently the most convenient to use.

First, we have to reduce the expressions (1) and (2) of Art. 41, namely,

$$\frac{\mu \cos. i'}{\cos. i} \cdot \frac{\cos. i'}{u'_1} = \left(\frac{\mu \cos. i'}{\cos. i} - 1\right)\frac{1}{r} - \frac{\cos. i}{u} \quad . \; . \; . \; . \; . \; (1)$$

$$\frac{\mu}{u'_2} = \left(\frac{\mu \cos. i'}{\cos. i} - 1\right)\frac{\cos. i}{r} - \frac{1}{u} \quad . \; . \; . \; . \; . \; (2)$$

to approximate expressions on the supposition that i and i' are small, and therefore that

$$i' = \frac{i}{\mu} \text{ nearly}$$

$$\cos. i = 1 - \frac{i^2}{2} \text{ nearly}$$

$$\cos. i' = 1 - \frac{i'^2}{2} \text{ nearly}$$

$$= 1 - \frac{i^2}{2\mu^2}$$

Substituting these values in (1), we have,

$$\frac{\mu}{u'_1} \cdot \frac{\left(1 - \frac{i^2}{2\mu^2}\right)^2}{\left(1 - \frac{i^2}{2}\right)} = \left(\frac{\mu\left(1 - \frac{i^2}{2\mu^2}\right)}{\left(1 - \frac{i^2}{2}\right)} - 1\right) \cdot \frac{1}{r} - \frac{1 - \frac{i^2}{2}}{u}$$

reducing, and noting that $\left(1 - \frac{i^2}{2}\right)^2 = (1 - i^2)$ nearly

$$\left(1 - \frac{i^2}{2\mu^2}\right)^{-2} = \left(1 + \frac{i^2}{\mu^2}\right) \text{ nearly}$$

we have

$$\frac{\mu}{u'_1}=\left\{\mu-1+\frac{i^2}{2}\left(1-\frac{1}{\mu}\right)\right\}\frac{1+\frac{i^2}{\mu^2}}{r}-\frac{(1-i^2)\left(1+\frac{i^2}{\mu^2}\right)}{u}$$

$$=\frac{\mu-1}{r}-\frac{1}{u}+i^2\left\{\left(\frac{\mu-1}{2\mu}+\frac{\mu-1}{\mu^2}\right)\frac{1}{r}+\frac{\mu^2-1}{\mu^2}\cdot\frac{1}{u}\right\}$$

$$=\frac{\mu-1}{r}-\frac{1}{u}+\frac{i^2(\mu-1)}{2\mu^2}\left\{\frac{\mu+2}{r}+\frac{2(\mu+1)}{u}\right\} \quad \ldots\ldots\ldots\ldots \quad (3)$$

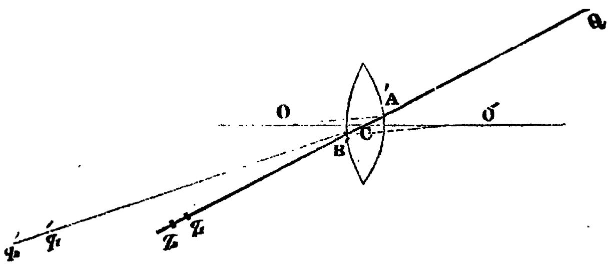
Again, substituting in (2), we have,

$$\frac{\mu}{u'_2}=\left\{\mu\left(1-\frac{i^2}{2\mu^2}\right)-\left(1-\frac{i^2}{2}\right)\right\}\frac{1}{r}-\frac{1}{u}$$

$$=\left\{\mu-1+\frac{i^2}{2}\left(\frac{\mu-1}{\mu}\right)\right\}\frac{1}{r}-\frac{1}{u}$$

$$=\frac{\mu-1}{r}-\frac{1}{u}+\frac{i^2(\mu-1)}{2\mu r} \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

Taking the same figure and letters as in the last proposition, we have at the second refraction



$$-\frac{\mu}{B'q_1}=\frac{\mu-1}{s}-\frac{1}{v_1}+\frac{i^2(\mu-1)}{2\mu^2}\left\{\frac{\mu+2}{s}+\frac{2(\mu+1)}{v_1}\right\}$$

$$-\frac{\mu}{B'q_2}=\frac{\mu-1}{s}-\frac{1}{v_2}+\frac{i^2(\mu-1)}{2\mu s}$$

But,
$$-\frac{\mu}{B'q'_1}=-\frac{\mu}{u'_1}-\left(\frac{t}{\mu}\sec. i'\right)\left(\frac{\mu}{u'_1}\right)^2$$
$$-\frac{\mu}{B'q'_2}=-\frac{\mu}{u'_2}-\left(\frac{t}{\mu}\sec. i'\right)\left(\frac{\mu}{u'_2}\right)^2$$

Substituting the values of $\frac{\mu}{u'_1}$ and $\frac{\mu}{u'_2}$ from (3) and (4) in the above, we have

$$\frac{1}{v_1}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}+$$
$$+\frac{i^2(\mu-1)}{2\mu^2}\left\{\frac{\mu+2}{r}+\frac{2(\mu+1)}{u}+\frac{\mu+2}{s}+\frac{2(\mu+1)}{v_1}\right\}$$
$$+\frac{t}{\mu}\left(1+\frac{i^2}{2\mu^2}\right)\left\{\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\right.$$
$$\left.+\frac{i^2(\mu-1)}{\mu^2}\left(\frac{\mu+2}{r}+\frac{2(\mu+1)}{u}\right)\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\right\}$$

or

$$\frac{1}{v_1}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+$$
$$+\frac{i^2(\mu-1)}{2\mu^2}\left\{(\mu+2)\left(\frac{1}{r}+\frac{1}{s}\right)+\right.$$
$$\left.+2(\mu+1)\left[\frac{1}{u}+(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2\right]\right\}$$
$$+\frac{i^2t}{\mu}\left\{\frac{1}{2\mu^2}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\frac{\mu-1}{\mu^2}\left(\frac{\mu+2}{r}+\frac{2(\mu+1)}{u}\right)\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\right\}$$

or

$$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2$$
$$+\frac{i^2(\mu-1)}{2\mu}\left\{(2\mu+1)\left(\frac{1}{r}+\frac{1}{s}\right)\right\}$$
$$+\frac{i^2t}{2\mu^3}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left\{\frac{2\mu^3+\mu-3}{r}+\frac{2\mu^2-3}{u}\right\}$$

Also

$$\frac{1}{v_2}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}+\frac{i^2(\mu-1)}{2\mu}\left(\frac{1}{r}+\frac{1}{s}\right)+$$
$$+\frac{t}{\mu}\left(1+\frac{i^2}{2\mu^2}\right)\left\{\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\frac{i^2(\mu-1)}{\mu r}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\right\}$$

or

$$\frac{1}{v_2}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\frac{i^2(\mu-1)}{2\mu}\left(\frac{1}{r}+\frac{1}{s}\right)$$
$$+\frac{i^2t}{2\mu^3}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left\{\frac{(\mu-1(2\mu+1)}{r}-\frac{1}{u}\right\}$$

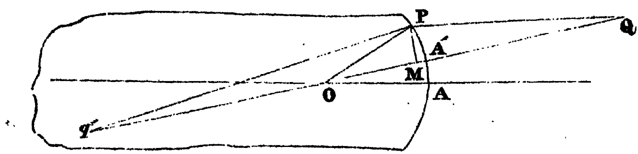
which give v_1 and v_2 as required. If we neglect the thickness in the small terms, we have for *very thin* lenses, the simple expressions.

$$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\frac{i^2(2\mu+1)}{2\mu f}$$
$$\frac{1}{v_2}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\frac{i^2}{2\mu f}$$

Making $i=0$, we have the primary and secondary foci coinciding in the small direct pencil, as at page 80, PART I.

ART. 44. PROP. *To investigate the aberration of a given ray of a diverging pencil which passes* obliquely *through a thin double convex lens, in the plane through the axis of the lens and the focus of the incident rays.*

When a pencil which is not very small passes through a lens obliquely, the positions of the foci are more conveniently connected



together when their distances are measured from the centers of

the curvatures of the surfaces, the approximations being obtained as in the case of direct pencils.

First, for a single refraction, as in the figure, let $QO=p$, $q'O=p'$, $OA'=r$, $A'M=x$, $PM=y$, and $y^2=2rx-x^2$.

As in former propositions

$$\frac{QP}{QO}=\frac{q'P}{\mu \, . \, q'O} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and
$$QP^2=QO^2+OP^2-2 \, . \, QO \, . \, OM$$
$$=p^2+r^2-2p(r-x)$$
$$=(p-r)^2\left(1+\frac{2px}{(p-r)^2}\right)$$

$$\therefore \quad QP=p-r+\frac{px}{p-r} \text{ nearly}$$

$$q'P^2=q'O^2+OP^2+2 \, . \, q'O \, . \, OM$$
$$=p'^2+r^2+2p'(r-x)$$
$$=(p'+r)^2\left(1-\frac{2p'x}{(p'+r)^2}\right)$$

$$\therefore \quad q'P=p'+r-\frac{p'x}{p'+r} \text{ nearly}$$

Substituting these values we have the fundamental equation (1) becoming as follows,

$$\frac{p-r}{p}+\frac{x}{p-r}=\frac{p'+r}{\mu p'}-\frac{x}{\mu(p'+r)}$$

dividing by r, &c.,

$$\frac{1}{\mu p'}=\left(1-\frac{1}{\mu}\right)\frac{1}{r}-\frac{1}{p}+\frac{x}{r}\left\{\frac{1}{p-r}+\frac{1}{\mu(p'+r)}\right\}$$

For the first approximation

$$\frac{1}{\mu p'}=\frac{\mu-1}{\mu r}-\frac{1}{p}$$

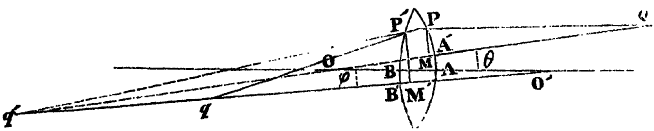
and substituting this first approximate value of $\mu p'$ in the small term we have

$$\frac{1}{\mu p'}=\frac{\mu-1}{\mu r}-\frac{1}{p}+\frac{x}{r}\left\{\frac{1}{p-r}+\frac{1}{\frac{\mu r p}{(\mu-1)p-\mu r}+\mu r}\right\}$$

$$=\frac{\mu-1}{\mu r}-\frac{1}{p}+\frac{x(\mu-1)}{\mu^2 r^2}\left(\frac{\mu r+p}{p-r}\right) \quad \ldots\ldots\ldots\ldots \quad (1)$$

When p is less than $\frac{\mu r}{\mu-1}$, we have q' on the same side of O with Q, and p' being taken negative is again given by the above equation.

Let the letters in the annexed figure, which are the same as in the last, refer to similar points, and let O' be the center of curvature of the second surface of which the radius is $O'B'=s$; let q



be the point where the emergent ray $P'q$ cuts $O'q'$ the direction of the ray which emerges perpendicularly to the second surface at B'; let $qO'=q$, $q'O'=q'$ and $B'M'=x'$.

At the first refraction we have the expression as before found, and at the second refraction, we have

$$-\frac{1}{\mu q'}=-\frac{1}{\mu \cdot q'O'}=\frac{\mu-1}{\mu s}-\frac{1}{q}+\frac{x'(\mu-1)}{\mu^2 s^2}\left(\frac{\mu s+q}{q-s}\right) \quad \ldots \quad (2)$$

To connect $q'O$ with $q'O'$, let $OO'=r+s-AB=d$, and let $\angle QOO'=\theta$; then

$$q'O'^2=q'O^2+OO'^2+2\,.\,q'O\,.\,OO'\cos.\theta$$

or $$q'^2=p'^2+d^2+2p'd\cos.\theta$$

Also, θ being supposed small, we have $\cos.\theta=1-\frac{\theta^2}{1.2}$ nearly

$$\therefore \quad q'^2 = (p'+d)^2 - 2p'd\frac{\theta^2}{1.2}$$

or
$$q' = p' + d - \frac{p'd}{p'+d}\cdot\frac{\theta^2}{1.2} \text{ nearly} \quad \ldots \quad \ldots \ldots \quad (3)$$

If in this equation (3) we substitute the value of $q'O'$ from (2) and of p' from (1) we have the relation required between p, q, r, s, and d, as follows

$$-q' = \cfrac{1}{\frac{\mu-1}{s} - \frac{\mu}{q} + \frac{x'(\mu-1)}{\mu s^2}\left(\frac{\mu s+q}{q-s}\right)}$$

$$= \frac{sq}{(\mu-1)q-\mu s}\left\{1 - \frac{x'(\mu-1)}{\mu s^2}\left(\frac{\mu s+q}{q-s}\right)\left(\frac{sq}{(\mu-1)q-\mu s}\right)\right\}$$

similarly

$$p' = \frac{rp}{(\mu-1)p-\mu r} - \frac{x(\mu-1)}{\mu r^2}\left(\frac{\mu r+p}{p-r}\right)\left(\frac{rp}{(\mu-1)p-\mu r}\right)^2$$

Substituting these values in (3), and using only the first approximate values of p' and q' in the small terms, also putting p'_1 and q'_1 for the first approximations of p' and q', which are known when p and q are known,

because
$$p'_1 = \frac{rp}{(\mu-1)p-\mu r}, \quad q'_1 = \frac{-sq}{(\mu-1)q-\mu s}$$

we have

$$-\frac{sq}{(\mu-1)q-\mu s} = p'_1 + d - \frac{p'_1 d\,.\,\theta^2}{(p'_1+d)^2} - \frac{\mu-1}{\mu}\left\{\frac{x}{r^2}\,\frac{(\mu r+p)p'^2_1}{(p-r)} + \frac{x'}{s^2}\,\frac{(\mu s+q)q'^2_1}{(q-s)}\right\}$$

Taking the reciprocals of the two sides of this equation, and treating the small terms in the usual approximate manner

$$\frac{1}{q} = \frac{\mu-1}{\mu s} + \frac{1}{\mu(p'_1+d)} + \frac{1}{\mu(p'_1+d)^2}\left\{\frac{\theta^2\,.\,d\,.\,p'_1}{2(p'_1+d)} + \frac{\mu-1}{\mu}\left\{\frac{xp'^2_1(\mu r+p)}{r^2(p-r)} + \frac{x'q'^2_1(\mu s+q)}{s^2(q-s)}\right\}\right\}$$

$$=\frac{1}{q_1}+\frac{\theta^2 d p'_1}{2\mu q'^3_1}+\frac{\mu-1}{\mu^2}\left\{x\left(\frac{p'_1}{q'_1}\right)^2\frac{(\mu r+p)}{r^2(p-r)}+\frac{x'(\mu s+q)}{s^2(q-s)}\right\}.$$

In this expression we have the small terms containing the quantities x, x' and θ; but when θ and x (or $y=PM$) are given then x' (or $y'=P'M'$) is known.

If we make $\theta=0$, and substitute for q its value $v+s$, for p, $u+r$, and for d, $r+s$ in a thin lens, we fall, after the reductions upon the result of Art. 34; and the above expression shews us that the aberration of an oblique pencil does not change rapidly from the value for the direct pencil whilst θ is very small.

To find the value of the aberration, q_1 being the first approximate value of q, or

$$\frac{1}{q_1}=\frac{\mu-1}{\mu s}+\frac{1}{\mu(p'_1+d)}$$

Then the longitudinal aberration, measured in the case of the figure towards the lens since $\frac{1}{q}>\frac{1}{q_1}$, is $q_1-q=q_1{}^2\left(\frac{1}{q}-\frac{1}{q_1}\right)$ nearly,

$$=\frac{q_1{}^2}{\mu(p'_1+d)^2}\left\{\frac{\theta^2 dp'_1}{2(p'_1+d)}+\frac{\mu-1}{\mu}\left\{\frac{xp'_1{}^2(\mu r+p)}{r^2(p-r)}+\frac{x'q'_1{}^2(\mu s+q_1)}{s^2(q_1-s)}\right\}\right\}.$$

If we examine the effect of the refractions in other planes, we find that all the rays falling on the first surface at the same distance from A' will have the same angle of incidence, and meet the line QOq' in the same point, but when they fall on the second surface, they will not be equally distant from B', nor have the same angle of incidence on the surface, and will not therefore be refracted again accurately in a conical pencil: when θ is very small, however, compared with the value of x (or y), the deviation from such a form will not be considerable.

When θ is larger, we obtain from experiment with an equiconvex lens, by receiving on a screen the light from a distant luminous point after it has passed through the lens, and at different distances, the *figures of oblique aberration* as follows:

where F would generally be considered the best focus, the light being strongly concentrated at the head of the figure, although there is a lengthened coma; but an image of an object formed by such foci is necessarily very indistinct.

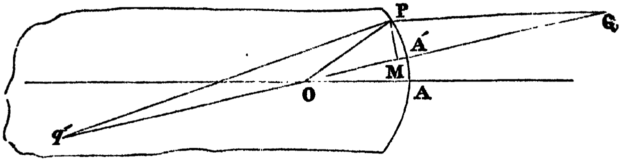
With the same lens and a pencil, either direct or but very little oblique, we have for comparison such a series, at different distances, as the following:

When the *lens* is *small* compared with the *obliquity*, we obtain such figures as the following:

where q_1 is the primary focal line, o the circle of confusion, and q_2 the secondary focal line. The circle of confusion as a focus will give only indistinct images of objects, when the obliquity is considerable.

Art. 45. Prop. *To put the expression for the* oblique aberration, *of the last article, into a form for application to particular cases.*

From the last article we have



the aberration

$$=\frac{q_1^2}{\mu(p'_1+d)^2}\left\{\frac{\theta^2 dp'_1}{2(p'_1+d)}+\frac{\mu-1}{\mu}\left(\frac{xp'^2_1(\mu r+p)}{r^2(p-r)}+\frac{x'q'^2_1(\mu s+q_1)}{s^2(q_1-s)}\right)\right\}$$

Let the angles which the edge of the lens subtends at O and O' respectively be α and α', and let $\angle\, OO'q=\phi$.

Then generally $x=\frac{PM^2}{2r}$, $x'=\frac{P'M'^2}{2s}$ nearly,

and for the upper edge of the lens in the figure, we have

$$x=\frac{r^2(\alpha-\theta)^2}{2r} \qquad x'=\frac{s^2(\alpha'+\phi)^2}{2s}$$

$$=\frac{r(\alpha-\theta)^2}{2} \qquad =\frac{s(\alpha'+\phi)^2}{2}$$

also $\frac{\sin.\,\theta}{\sin.\,\phi}=\frac{q'_1}{p'_1}=\frac{\theta}{\phi}$ nearly, and $q'_1=p'_1+d$, $r\alpha=s\alpha'$.

$$\therefore \text{ the aberration}=\frac{q_1^2}{\mu}\left\{\frac{\theta^2 dp_1}{2q'^3_1}+\right.$$

$$\left.+\frac{\mu-1}{2\mu}\left((\alpha-\theta)^2\cdot\frac{p'^2_1}{q'^2_1}\cdot\frac{(\mu r+p)}{r(p-r)}+(\alpha'+\phi)^2\cdot\frac{\mu s+q_1}{s(q_1-s)}\right)\right\}$$

For the lower edge, as in the figure, we must put $-\alpha$ for α and $-\alpha'$ for α', whilst θ and ϕ remain the same.

For a pencil of *parallel* rays falling obliquely on *thin convex* lenses of different forms, we find the values of the aberration for the *full aperture* as in the following table.

A TABLE OF THE VALUES OF THE OBLIQUE ABERRATION FOR DIFFERENT FORMS OF CONVEX LENSES WITH *parallel* INCIDENT RAYS.

Species of Lens.	The aberration; the upper sign of the middle term referring to the upper edge of the lens and the lower sign to the lower edge.	Focal length.
Equi-convex . . .	$\frac{5}{3}\alpha^2 r \pm \frac{8}{3}\alpha\phi r + \frac{7}{3}\phi^2 r$	r
Crossed-lens . . .	$\frac{5}{8}\alpha^2 r \pm \frac{9}{14}\alpha\phi r + \frac{351}{14}\phi^2 r$	$\frac{12}{7}r$
Crossed-lens inverted	$\frac{115}{12}\alpha^2 r \pm \frac{1007}{252}\alpha\phi r + \frac{1121}{3024}\phi^2$	$\frac{2}{7}r$
Plano-convex . . .	$\frac{9}{4}\alpha^2 s \pm \frac{9}{2}\alpha\phi s + \frac{9}{4}\phi^2 s$	$2s$
Convexo-plane . . .	$\frac{7}{12}\alpha^2 r \mp \frac{1}{3}\alpha\phi r + \frac{5}{3}\phi^2 r$	$2r$

In the above table the *first* terms are the aberration for a *direct* pencil, and when brought to the same form coincide with the results of the table at page 86.

We see that the effect of obliquity is different with different forms of lenses, and by the general expression we see that as θ and ϕ have different signs, a form may be assigned to the lens such that the term depending on the first powers of θ and ϕ may vanish.

If we take ψ for the angle which the axis of the oblique pencil, passing through the center of the lens, makes with the line OO' the axis of the *thin* lens; and u r, as before used, we have

$$\frac{\sin.\ \phi}{\sin.\ \psi} = \frac{v}{q_1} = \frac{\phi}{\psi} \text{ nearly,}$$

$$\frac{\sin.\ \theta}{\sin.\ \psi} = \frac{u}{p} = \frac{\theta}{\psi} \text{ nearly.}$$

By the method before used we find

$$p=u+r-\frac{ru}{2(u+r)}\psi^2$$

and substituting in the values of p'_1 and q'_1, we arrive at an equation

$$\frac{1}{v}=(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)-\frac{1}{u}+a\alpha^2+b\alpha\psi+c\psi^2$$

where a, b and c are functions of μ, u, r and s; but the refracted pencil not being symmetrical with respect to any line, it is unnecessary to discuss this expression for $\frac{1}{v}$.

ART. 46. PROP. *To investigate the changes in the oblique aberration of lenses, arising from changes in the position of the conjugate foci and from changes in the forms of the lenses.*

Taking the lens double convex and the obliquity small, with the ray emerging perpendicularly to the second surface for the axis of the refracted pencil, we must examine the changes in the coefficient of the first power of ϕ, under different circumstances.

The term in the aberration with θ and ϕ is

$$\frac{q_1^2}{\mu}\left(\frac{\mu-1}{2\mu}\right)\left\{2\alpha'\phi\ \frac{\mu s+q_1}{s(q_1-s)}-2\alpha\theta\left(\frac{p'_1}{q'_1}\right)^2\cdot\frac{\mu r+p}{r(p-r)}\right\}$$

$$=\frac{q_1^2(\mu-1)}{\mu^2}\phi\alpha\left(\frac{r}{s}\cdot\frac{\mu s+q)}{s(q_1-s)}-\frac{p'_1}{q'_1}\cdot\frac{\mu r+p}{r(p-r)}\right)$$

and we have to examine the value and sign of the coefficient

$$\frac{r}{s}\cdot\frac{\mu s+q_1}{s(q_1-s)}-\frac{p'_1}{q'_1}\cdot\frac{\mu r+p}{r(p-r)}$$

Putting, as at page 81, $mu=f$, $nr=f$, we have

$$\frac{1}{v}=\frac{1-m}{f}$$

$$q_1=s+v\ , \qquad\qquad p=u+r\ ,$$

$$\frac{1}{s}=\frac{1-n(\mu-1)}{(\mu-1)f}$$

$$\frac{r}{s}=\frac{1-n(\mu-1)}{n(\mu-1)}$$

$$\frac{p'_1}{q'_1}=\frac{u'-r}{u'+s}$$

$$=\frac{(m+n)[1-n(\mu-1)]}{n[\mu-(m+n)(\mu-1)]}$$

$$\frac{\mu s+q_1}{s(q_1-s)}=\frac{(\mu+1)s+v}{sv}$$

$$=\frac{\mu+1}{v}+\frac{1}{s}$$

$$=\frac{\mu^2-(\mu-1)[(\mu+1)m+n]}{(\mu-1)f}$$

$$\frac{\mu r+p}{r(p-r)}=\frac{\mu+1}{u}+\frac{1}{r}$$

$$=\frac{(\mu+1)m+n}{f}$$

substituting these quantities in the coefficient and reducing, we find that the term in the aberration will be greater than, equal to or less than zero, as

$$\mu^2-m(2\mu^2-\mu-1)-n(\mu^2-1)>=<0$$

and when equal to zero, we have

$$n=\frac{\mu^2-m(2\mu^2-\mu-1)}{\mu^2-1}$$

Ex. 1. To find the form of the lens such that the oblique aberration may be the same as the direct for parallel incident rays, or the above expression $=0$.

We have now $m=0$ since the incident rays are parallel,

$$\therefore\ n=\frac{\mu^2}{\mu^2-1}$$

and if $\mu=\frac{3}{2}$, $r=\frac{f}{n}=\frac{5}{9}f$

$$s=\frac{(\mu-1)f}{1-n(\mu-1)}=5f$$

$$\therefore\ s=9r$$

This accords with the results of the table, as we see that the sign of the coefficient changed in passing from the crossed lens to the convexo-plane.

Ex. 2. To find the positions of the conjugate foci of an equi-convex lens when the effect of obliquity vanishes.

We have now $n=1$, since $r=s=f$,

and $$m=\frac{1}{2\mu^2-\mu-1}$$

if $\mu=\frac{3}{2}$, then $m=\frac{1}{2}$

and $$u=\frac{f}{m}=2f=v,$$

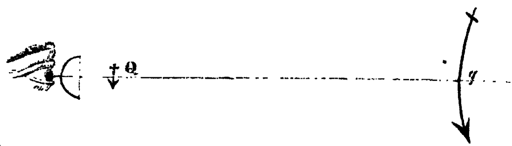
or the conjugate foci are at equal distances from the lens. When the luminous point is nearer or further than this distance from the lens the coefficient changes sign.

Ex. 3. To find the effect of obliquity when a virtual image of a luminous point is formed by an equi-convex lens.

We have now u less than f, and therefore m greater than unity, and the coefficient of ϕ is negative, as it had been when the distance of the luminous point was between f and $2f$ and the image real. The direct aberration is now, however, measured from the lens beyond the first approximate focus, as shewn at page 93.

Ex. 4. To find the form of the lens of the *single* microscope such that the oblique aberration may be the same as the direct, or such that the coefficient of ϕ is zero.

The eye being supposed close to the lens, we have v equal to the least distance



of distinct vision nearly; and if the magnifying power $=a=\frac{v}{u}$ the focal length of the lens is known.

But
$$-\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$
$$=-\frac{1}{au}$$

and
$$u=f\cdot\frac{a-1}{a}$$
$$\therefore\ m=\frac{a}{a-1}$$

and the equation becomes

$$\mu^2-\frac{a}{a-1}(2\mu^2-\mu-1)+n(\mu^2-1)=0$$
$$\therefore\ n=-\frac{\mu^2+a(\mu^2-\mu-1)}{(\mu^2-1)(a-1)}$$

Let $a=10$, and $\mu=\frac{3}{2}$, we have

$$n=\frac{1}{45}$$

and $$r=45f,\qquad s=\frac{45}{89}f$$

and the lens is a plano-convex very nearly.

Ex. 5. To find the form of the lens, so that the oblique aberration may be the same as the direct when a *virtual* image is magnified *two, three,* and *four* times.

We have now to put a equal to two, or three, or four, in the expression of the last example, when $\mu=\frac{3}{2}$.

$$n=-\frac{\mu^2+a(\mu^2-\mu-1)}{(\mu^2-1)(a-1)}$$

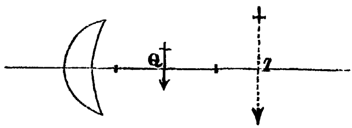
$$=\frac{a-9}{5\,(a-1)}$$

If $a=2$, then $n=-\frac{7}{5}$

$$r=\frac{f}{n}=-\frac{5}{7}f$$

$$s=\frac{f}{2-n}=+\frac{5}{17}f$$

and the lens and image are as in the figure, drawn to the scale $f=1$ inch.

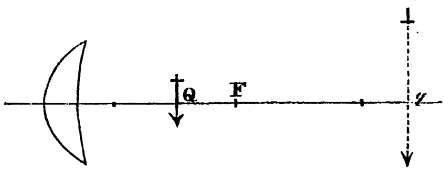


If $a=3$, then $n=-\frac{3}{5}$

$$r=\frac{f}{n}=-\frac{5}{3}f$$

$$s=\frac{f}{2-n}=\frac{5}{13}f$$

the lens and image being as in the figure, drawn to $f=1$ inch, as before.



If $a=4$, then $n=-\frac{1}{3}$

$$r=\frac{f}{n}=-3f$$

$$s=\frac{f}{2-n}=\frac{3}{7}f$$

and the lens is a meniscus, but the first surface of long radius.

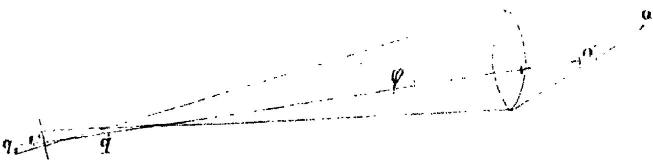
When the image is magnified *nine* times, the first surface is plane.

These Examples will be found referred to in Art. 63 in discussing the forms of the achromatic lenses, used as powers for the microscope.

ART. 47. PROP. *To find whether* the coma *in given cases is to or from the axis of the lens.*

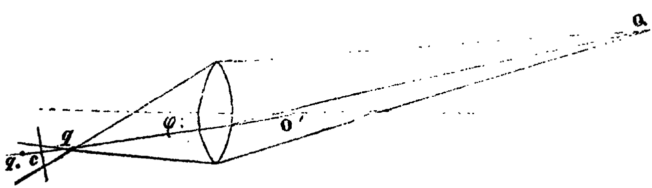
We have to trace the effect of the term with $\alpha\phi$ in the oblique aberration, and will suppose the lens to be equi-convex.

First, let the luminous point Q be distant between f and $2f$ from the lens, and therefore the image real, as in the figure. If q_1 be the first approximate focus, $q_1 q$ the aberration for a direct pencil, the ray refracted at the upper edge of the lens cuts $O'q_1$

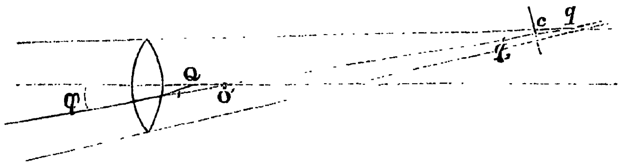


between q_1 and q, because the term with $\alpha\phi$, from Examples 2 and 3, is then negative; and similarly the ray refracted at the lower edge cuts $O'q_1$ between O' and q as in the figure. If we now take a section of the pencil at c we have around that point a condensed nucleus of light, and *a coma* turned towards the axis of the lens.

Secondly, let Q be beyond the distance $2f$, and the image is still real as in the figure. We have the term with $\alpha\phi$ producing an opposite effect to that of the last case, and as shewn in the figure, the coma is now turned from the axis.



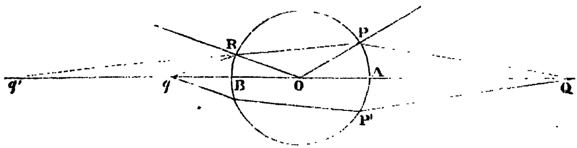
Thirdly, let Q be nearer to the lens than the focal length f and therefore the image virtual. The direct aberration $q_1 q$ is now measured as in the next figure, beyond q_1 and the term with $\alpha\phi$,



as shewn in Example 3, being negative for the upper edge and positive for the lower, we have the coma in a section at c turned towards the axis.

ART. 48. PROP. *To find the form of the pencil which has been refracted directly by a sphere, to a second approximation.*

The sphere as described in Arts. 75 and 82, PART I, has gene-



rally a limited aperture at the center. Let z be the radius of this aperture, and let y and y' be the ordinates at P and R respectively, as in the figure. Then for the extreme rays which pass through the aperture we have z a mean proportional between y and y' nearly. Using the notation of Art. 75, PART I.,

$$QO=p,\ q'O=q',\ qO=q,\ AO=r,$$

we have
$$z=\frac{1}{2}(y+y')=\frac{yq'}{q'+r} \text{ nearly}$$
$$=\frac{y'q'}{q'-r}$$

or using the first approximate value of q'

we have
$$y=zr\left(\frac{1}{r}+\frac{1}{q'}\right)=z\mu r\left(\frac{1}{r}-\frac{1}{p}\right)$$

Also putting $y^2 = 2rx$ in the results of Art. 44,

$$QP = p - r + \frac{y^2}{2r}\left(\frac{p}{p-r}\right)$$

$$q'P = q' + r - \frac{y^2}{2r}\left(\frac{q'}{q'+r}\right)$$

and substituting in $\quad \dfrac{QP}{QO} = \dfrac{q'P}{\mu \cdot q'O}$

we have $\displaystyle \frac{1}{\mu q'} = \frac{\mu-1}{\mu r} - \frac{1}{p} + \frac{y^2}{2r}\left(\frac{1}{p-r} + \frac{1}{\mu(q'+r)}\right)$

$$= \frac{\mu-1}{\mu r} - \frac{1}{p} + \frac{y^2}{2r}\left(\frac{p}{p-r}\right)^2 \frac{\mu-1}{\mu}\left(\frac{1}{p} + \frac{1}{\mu r}\right)$$

$$= \frac{\mu-1}{\mu r} - \frac{1}{p} + \frac{z^2\mu(\mu-1)}{2r}\left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{p} + \frac{1}{\mu r}\right)$$

Supposing the pencil to be incident in the contrary direction, diverging from q, we have

$$\frac{1}{\mu q'} = \frac{\mu-1}{\mu r} - \frac{1}{q} + \frac{z^2\mu(\mu-1)}{2r}\left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{q} + \frac{1}{\mu r}\right)$$

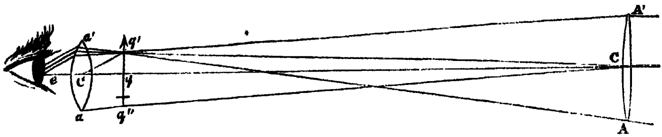
and adding

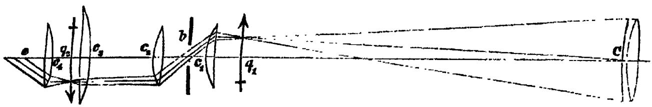
$$\frac{1}{q} = \frac{2(\mu-1)}{\mu r} - \frac{1}{p} + \frac{z^2\mu(\mu-1)}{2r}\left\{\left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{p} + \frac{1}{\mu r}\right) + \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{q} + \frac{1}{\mu r}\right)\right\}$$

as required.

ON EXCENTRICAL REFRACTION.

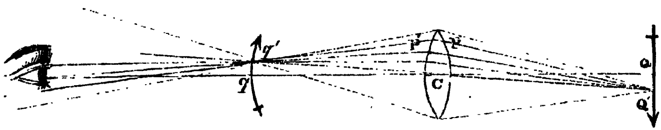
In the preceding Propositions the axis of the pencil is supposed to pass through or nearly through the center of the lens, and the pencil is called a *centrical direct* or *oblique* pencil accordingly; the pencils passing through the object-glasses of telescopes are always one or other of these, but when they reach the eye-piece the circumstances are changed, and every pencil except the direct one is then refracted *excentrically*. We see how this arises in the passage through the eye-lenses of telescopes by referring to the figures of Arts. 95 and 97, PART I., as repeated below, and the same occurs in the reflecting telescopes and microscopes.

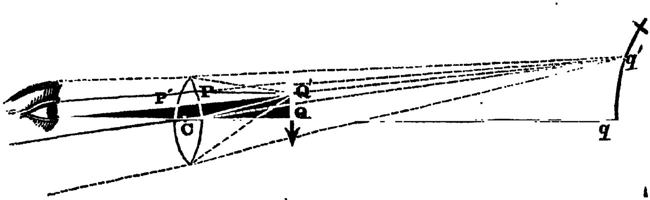




If we examine the figure of the simple astronomical telescope, we see that C the center of the object-glass is a fixed point through which the axis of every pencil passes; and the point e where it meets the axis again is the focus conjugate to C for the eye-glass, and may be taken as also a fixed point when the aberration at the eye-glass can be neglected or when the point q' in the image is near q. The points C and e being known, the circumstances of the excentrical refraction can be discussed. There are again like points at C, b, and e in the telescope with the erecting eye-tube, in the lower figure.

Whenever we see the image, real or virtual, of an object formed by a single lens, and the eye is at some distance from the lens, the vision takes place by excentrical pencils, as in the figures, where

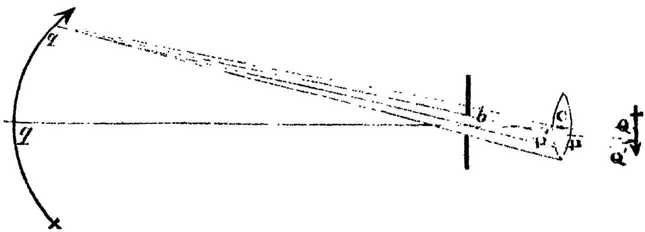




the upper one represents a real image qq', seen in the air, of an object QQ'; and the lower one a virtual image qq' of an object QQ', seen by the eye at e.

Of the rays, which diverge from any point Q' in an object, fall upon the lens and converge to, or diverge from, an approximate focus q', only the small excentrical pencil $Q'PP'e$ enters the eye. The place e of the optical center of the eye, or nearly the center of curvature of the cornea, is now the fixed point that determines the excentrical pencil by which the image is seen, which can be determined when the positions of the given lens, the eye and the point Q' in the object are known.

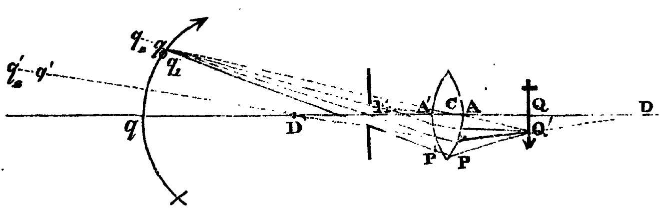
In many common microscopes a diaphragm is placed behind the object-glass, which limits the magnitude of all the pencils and causes all points in the image except the central one to be formed by excentrical pencils, as in the figure. If C be the center of the

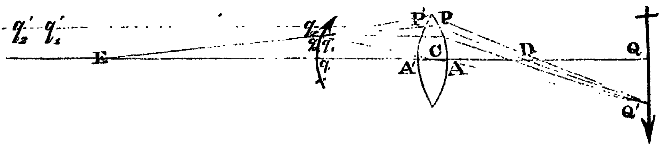


lens, b the aperture in the diaphragm, then q' the point in the image conjugate to Q' in the object will be formed by the excentrical pencil $Q'PP'q'$; but it will be nearly in the line drawn from Q' through C. In this case b is the fixed point which determines the excentrical pencils forming the image.

Art. 49. Prop. *To find the primary and secondary foci of* a small excentrical *pencil, which coming from a given luminous point, crosses the axis of the lens at a given point.*

Let Q' in the figures be the luminous point, and q_0 its approximate real image, which is the circle of least confusion between q_1





and q_2, the primary and secondary foci, and $q'_1\, q'_2$ be the corresponding foci after the first refraction.

Let E be the point where the axis of the emergent pencil cuts the axis of the lens, and D that where the axis of the incident pencil cuts it; $Q'PP'q_0$ being the path of the axis.

Let r and s be the radii of the first and second surfaces of the lens; t the thickness AA', which we suppose small compared with the focal length; $Q'P=u$, $q'_1P=u'_1$, $q'_2P=u'_2$, $q_1P'=v_1$, $q_2P'=v_2$; let i and e be the angles of incidence and emergence, i' and e' the corresponding angles respectively, within the lens.

From Art. 41 we have

$$\frac{\mu\cos. i'}{\cos. i}\cdot\frac{\cos. i'}{u'_1}=\left(\frac{\mu\cos. i'}{\cos. i}-1\right)\frac{1}{r}-\frac{\cos. i}{u}$$

$$\frac{\mu}{u'_2}=\left(\frac{\mu\cos. i'}{\cos. i}-1\right)\frac{\cos. i}{r}-\frac{1}{u}$$

At the second refraction we have

$$-\frac{\mu\cos. e'}{\cos. e}\cdot\frac{\cos. e'}{q'_1P'}=\left(\frac{\mu\cos. e'}{\cos. e}-1\right)\frac{1}{s}-\frac{\cos. e}{v_1}$$

$$-\frac{\mu}{q'_2P'}=\left(\frac{\mu\cos. e'}{\cos. e}-1\right)\frac{\cos. e}{s}-\frac{1}{v_2}$$

But,
$$\frac{1}{q'_1P'}=-\frac{1}{u'_1}-\frac{PP'}{u'^2_1}$$
$$-\frac{1}{q'_2P'}=-\frac{1}{u'_2}-\frac{PP'}{u'^2_2}$$

Therefore, substituting the values in these expressions, we have

$$-\left(\frac{\mu\cos e'}{\cos. e}-1\right)\frac{\cos. e}{\cos.^2 e'}\cdot\frac{1}{s}+$$
$$+\frac{\cos.^2 e}{\cos.^2 e'}\cdot\frac{1}{v_1}=\left(\frac{\mu\cos. i'}{\cos. i}-1\right)\frac{\cos. i}{\cos.^2 i'}\cdot\frac{1}{r}-\frac{\cos.^2 i}{\cos.^2 i'}\cdot\frac{1}{u}+\frac{\mu\,.\,PP'}{u'^2_1}$$

and $-\left(\frac{\mu\cos. e'}{\cos. e}-1\right)\frac{\cos. e}{s}+\frac{1}{v_2}=\left(\frac{\mu\cos. i'}{\cos. i}-1\right)\frac{\cos. i}{r}-\frac{1}{u}+\frac{\mu\,.\,PP'}{u'^2_2}$

which give

$$\frac{1}{v_1}=\left\{\left(\frac{\mu\cos. i'}{\cos. i}-1\right)\frac{\cos. i}{\cos.^2 i'}\cdot\frac{1}{r}+\left(\frac{\mu\cos. e'}{\cos. e}-1\right)\frac{\cos. e}{\cos.^2 e'}\cdot\frac{1}{s}-\right.$$
$$\left.-\frac{\cos.^2 i}{\cos.^2 i'}\cdot\frac{1}{u}\right\}\frac{\cos.^2 e'}{\cos.^2 e}+\frac{\mu\,.\,PP'}{u'^2_1}\cdot\frac{\cos.^2 e'}{\cos.^2 e}$$

$$\frac{1}{v_2}=\left(\frac{\mu\cos. i'}{\cos. i}-1\right)\frac{\cos. i}{r}+\left(\frac{\mu\cos. e'}{\cos. e}-1\right)\frac{\cos. e}{s}-\frac{1}{u}+\frac{\mu\,.\,PP'}{u'^2_2}$$

We perceive an analogy between these expressions and those found in Art. 29 for the passage of a small pencil through a prism, with which indeed they coincide, except as to the term involving PP' the thickness of the lens traversed, which is here only approximate, when we make r and s infinite for plane surfaces.

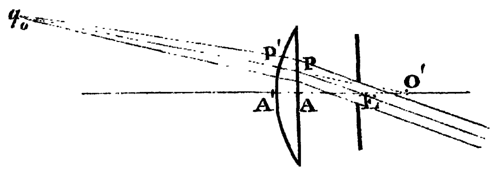
If the lens is of such a form that i and e are equal, the above are reduced to

$$\frac{1}{v_1}=\left(\frac{\mu\cos. i'}{\cos. i}-1\right)\left(\frac{1}{r}+\frac{1}{s}\right)\sec. i-\frac{1}{u}+\frac{\mu\,.\,PP'}{u'^2_1}\cdot\frac{\cos.^2 i'}{\cos.^2 i}$$
$$\frac{1}{v_2}=\left(\frac{\mu\cos. i'}{\cos. i}-1\right)\left(\frac{1}{r}+\frac{1}{s}\right)\cos. i-\frac{1}{u}+\frac{\mu\,.\,PP'}{u'^2_2}$$

which agree with those for the centrical pencil, except the term with PP'.

COR. There is one simple case in which the primary and secondary foci are easily shewn to coincide, and therefore the excentrical pencils are as accurate as the direct one.

Let the lens be plano-convex, or $r =$ infinity, and the incident pencil consist of parallel rays, or $u =$ infinity, and therefore u'_1 and $u'_2 =$ infinity, let also the diaphragm be placed at E so that O' being the center of curvature of the second surface, we have $AO' = \mu . AE$; then by Art. 28, PART I.,



the axis of the refracted pencil of parallel rays will pass in the direction of the radius $O'PP'q_0$, and the angles of incidence on the second surface and of emergence are each $=0$, and the expressions above are reduced to

$$\frac{1}{v_1} = \frac{(\mu - 1)}{s}$$
$$\frac{1}{v_2} = \frac{\mu - 1}{s}$$

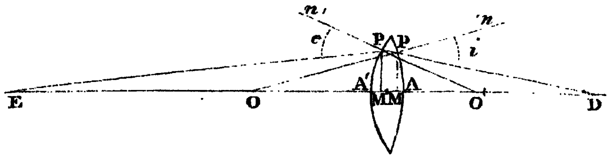
the result of Art. 52, PART I.

ART. 50. PROP. *To find the form of the lens such that a system of excentrical pencils shall have the angles of incidence on the first surface equal to those of emergence from the second.*

Although the difference between the values of v_1 and v_2 as found in the last proposition may be made in many cases to vanish, yet it will be generally desirable to know the form of the lens in which the confusion of the excentrical pencils shall not exceed that of centrical ones for the same angle of incidence.

Let $DPP'E$ be the course of the axis of a pencil which crosses the axis of the lens at D and E; let O and O' be the centers of curvature of the surfaces, $OP = r$, $O'P' = s$, $AD = c$, $A'E = b$.

Then OPn being the radius through P produced, the angle of in-



cidence $i = \angle DPn$, and similarly the angle of emergence $e = \angle EP'n'$; these are respectively the exterior angles of the triangles ODP, and $O'EP'$. Let $P'M' = y = PM$ nearly the distance of P or P' from the axis of the lens.

We have $$\angle i = \angle PDO + \angle POD = \frac{y}{c} + \frac{y}{r} \text{ nearly}$$

$$\angle e = \angle P'EO' + \angle P'O'E = \frac{y}{b} + \frac{y}{s} \quad \ldots\ldots$$

Or when $i = e$, we have,

$$\frac{1}{b} + \frac{1}{s} = \frac{1}{c} + \frac{1}{r}$$

Also $$\frac{1}{c} = \frac{1}{f} - \frac{1}{b}$$

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right)$$

Substituting for s and c, we have

$$\frac{1}{r} = \frac{2-\mu}{2(\mu-1)} \cdot \frac{1}{f} + \frac{1}{b}$$

and, $$\frac{1}{s} = \frac{\mu}{2(\mu-1)} \cdot \frac{1}{f} - \frac{1}{b}$$

which give r and s when the focal length of the lens and the place where the axis of the pencil crosses the axis of the lens are known.

Ex. 1. Required the form of the eye-glass of the astronomical telescope, so that the axis of each visual pencil shall have its angles of incidence and emergence very nearly equal.

We have now $b=f$ nearly, which gives

$$r=\frac{2}{3}f$$

$$s=2f$$

or the lens should be flattest on the side next the eye.

Ex. 2. Required the forms of the lenses, to make an eye-piece subject to the above rule for each lens, but in other respects similar to Huygens's eye-piece.

The field-glass will evidently be of the form of the last Example, say $r_1=\frac{2}{3}f_1$, $s_1=2f_1$.

If f_2 be the focal length of the eye-lens, then the distance between the lenses is $2f_2$ and the focus of the field-glass falls at the distance $f_1-2f_2=f_2$ from the eye-glass, and

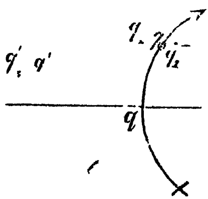
$$\frac{1}{b}=\frac{1}{f_2}+\frac{1}{f_2}=\frac{2}{f_2}$$

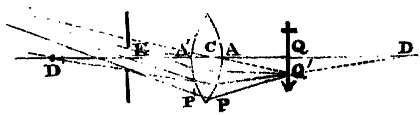
$\therefore$ we have from the formula $r_2=\frac{2}{5}f_2$, $s_2=-2f_2$

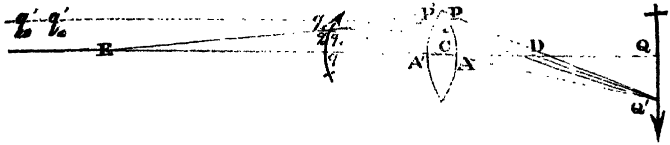
and the eye-lens will be a meniscus with the concave surface to the eye.

Art. 51. Prop. *To find the* approximate *expressions for the primary and secondary foci, when a* small excentrical *diverging pencil crosses, with* small obliquity, *the axis of the lens at a* given point.

Taking the figures and notation of Arts. 41 and 42, let also the angle $P'EA'=\theta=\frac{y}{b}$ nearly.







Also from Art 50,

$$i=y\left(\frac{1}{r}+\frac{1}{c}\right)=\theta\,.\,b\left(\frac{1}{r}+\frac{1}{c}\right)$$

$$e=y\left(\frac{1}{s}+\frac{1}{b}\right)=\theta\,.\,b\left(\frac{1}{s}+\frac{1}{b}\right)$$

The approximate expressions for the refraction at the first surface, from Art. 43, are

$$\frac{\mu}{u'_1}=\frac{\mu-1}{r}-\frac{1}{u}+\frac{i^2(\mu-1)}{2\mu^2}\left\{\frac{\mu+2}{r}+\frac{2(\mu+1)}{u}\right\}$$

$$\frac{\mu}{u'_2}=\frac{\mu-1}{r}-\frac{1}{u}+\frac{i^2(\mu-1)}{2\mu r}$$

And at the second refraction

$$-\frac{\mu}{q'_1P'}=-\frac{\mu}{u'_1}-\frac{\mu\,.\,PP'}{u'^2_1}$$

$$=\frac{\mu-1}{s}-\frac{1}{v_1}+\frac{e^2(\mu-1)}{2\mu^2}\left\{\frac{\mu+2}{s}+\frac{2(\mu+1)}{v_1}\right\}$$

$$-\frac{\mu}{q'_2P'}=-\frac{\mu}{u'_2}-\frac{\mu PP'}{u'^2_2}$$

$$=\frac{\mu-1}{s}-\frac{1}{v_2}+\frac{e^2(\mu-1)}{2\mu s}$$

To substitute for PP', let the central thickness $AA'=t$, and let the direction of the ray whilst within the lens cross the axis at a point D' which is given by the equation

$$-\frac{\mu}{A'D'}=\frac{\mu-1}{s}-\frac{1}{b}$$

then $\quad \sec. P'D'A = 1 + \frac{y^2}{2A'D'^2} = 1 + \frac{\theta^2 b^2}{2\mu^2}\left(\frac{1}{b} - \frac{\mu-1}{s}\right)^2$ nearly

and $\quad PP' = MM' \sec. P'D'A' = \left(t - \frac{y^2}{2r} - \frac{y^2}{2s}\right) \sec. P'D'A'$

$$= \left\{t - \frac{\theta^2 b^2}{2}\left(\frac{1}{r} + \frac{1}{s}\right)\right\} \cdot \left\{1 + \frac{\theta^2 b^2}{2\mu^2}\left(\frac{1}{b} - \frac{\mu-1}{s}\right)^2\right\}$$

$$= t - \frac{\theta^2 b^2}{2}\left\{\frac{1}{r} + \frac{1}{s} - \frac{t}{\mu^2}\left(\frac{1}{b} - \frac{\mu-1}{s}\right)^2\right\} \text{ nearly.}$$

Also $\quad \left(\frac{\mu}{u'_1}\right)^2 = \left(\frac{\mu-1}{r} - \frac{1}{u}\right)^2 +$

$$+ \frac{i^2(\mu-1)}{\mu^2}\left(\frac{\mu-1}{r} - \frac{1}{u}\right)\left\{\frac{\mu+2}{r} + \frac{2(\mu+1)}{u}\right\} \text{nearly}$$

$$\left(\frac{\mu}{u'_2}\right)^2 = \left(\frac{\mu-1}{r} - \frac{1}{u}\right)^2 + \frac{i^2(\mu-1)}{\mu r}\left(\frac{\mu-1}{r} - \frac{1}{u}\right)$$

Substituting the values of these quantities, we have,

$$\frac{1}{v_1} = (\mu-1)\left\{\frac{1}{r} + \frac{1}{s}\right\} - \frac{1}{u} + \frac{(\mu-1)}{2\mu^2}\left\{i^2\left(\frac{\mu+2}{r} + \frac{2(\mu+1)}{u}\right) + \right.$$

$$\left. + e^2\left(\frac{\mu+2}{s} + \frac{2(\mu+1)}{v_1}\right)\right\}$$

$$+ \left[\frac{t}{\mu} - \frac{\theta^2 b^2}{2\mu}\left\{\frac{1}{r} + \frac{1}{s} - \frac{t}{\mu^2}\left(\frac{1}{b} - \frac{\mu-1}{s}\right)^2\right\}\right]\left[\left(\frac{\mu-1}{r} - \frac{1}{u}\right)^2 + \right.$$

$$\left. + \frac{i^2(\mu-1)}{\mu^2}\left(\frac{\mu-1}{r} - \frac{1}{u}\right)\left(\frac{\mu+2}{r} + \frac{2(\mu+1)}{u}\right)\right]$$

$$\frac{1}{v_2} = (\mu-1)\left\{\frac{1}{r} + \frac{1}{s}\right\} - \frac{1}{u} + \frac{\mu-1}{2\mu}\left\{\frac{i^2}{r} + \frac{e^2}{s}\right\}$$

$$+ \left[\frac{t}{\mu} - \frac{\theta^2 b^2}{2\mu}\left\{\frac{1}{r} + \frac{1}{s} - \frac{t}{\mu^2}\left(\frac{1}{b} - \frac{\mu-1}{s}\right)^2\right\}\right]\left[\left(\frac{\mu-1}{r} - \frac{1}{u}\right)^2 + \right.$$

$$\left. + \frac{i^2(\mu-1)}{\mu r}\left(\frac{\mu-1}{r} - \frac{1}{u}\right)\right]$$

Or, with the small terms expressed with θ,

$$\frac{1}{v_1}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+$$

$$+\frac{\theta^2b^2(\mu-1)}{2\mu^3}\left\{\left(\frac{1}{r}+\frac{1}{c}\right)^2\left(\frac{\mu+2}{r}+\frac{2(\mu+1)}{u}\right)+\right.$$

$$\left.+\left(\frac{1}{s}+\frac{1}{b}\right)^2\left(\frac{\mu+2}{s}+\frac{2(\mu+1)}{v_1}\right)\right\}$$

$$-\frac{\theta^2b^2}{2\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left\{\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left(\frac{1}{r}+\frac{1}{s}\right)-\right.$$

$$-\frac{2t}{\mu}\left\{\frac{\mu-1}{\mu}\left(\frac{1}{r}+\frac{1}{c}\right)^2\left(\frac{\mu+2}{r}+\frac{2(\mu+1)}{u}\right)+\right.$$

$$\left.\left.+\frac{1}{2\mu}\left(\frac{1}{b}-\frac{\mu-1}{s}\right)^2\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\right\}\right\}$$

$$\frac{1}{v_2}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+$$

$$+\frac{\theta^2b^2(\mu-1)}{2\mu}\left\{\frac{1}{r}\left(\frac{1}{r}+\frac{1}{c}\right)^2+\frac{1}{s}\left(\frac{1}{s}+\frac{1}{b}\right)^2\right\}$$

$$-\frac{\theta^2b^3}{2\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left\{\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left(\frac{1}{r}+\frac{1}{s}\right)-\right.$$

$$\left.-\frac{2t}{\mu}\left\{\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left(\frac{1}{b}-\frac{\mu-1}{s}\right)^2\frac{1}{2\mu}+\frac{(\mu-1)}{r}\left(\frac{1}{r}+\frac{1}{c}\right)^2\right\}\right\}$$

If in these expressions we suppose t to be so small compared with r, s, &c., that we may omit the parts of the small terms which contain it, we have, after substituting the first approximate value of v_1 in the small term,

$$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+$$

$$\theta^2b^2\left\{\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{c}\right)^2\left(\frac{\mu+2}{r}+\frac{2(\mu+1)}{u}\right)+\right.\right.$$

$$\left.+\left(\frac{1}{s}+\frac{1}{b}\right)^2\left(\frac{2(\mu^2-1)}{r}+\frac{\mu(2\mu+1)}{s}-\frac{2(\mu+1)}{u}\right)\right\}-$$

$$\left.\frac{1}{2\mu}\left(\frac{1}{r}+\frac{1}{s}\right)\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2\right\}$$

$$\frac{1}{v_2}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+$$

$$\theta^2 b^2\left\{\frac{\mu-1}{2\mu}\left\{\frac{1}{r}\left(\frac{1}{r}+\frac{1}{c}\right)^2+\frac{1}{s}\left(\frac{1}{s}+\frac{1}{b}\right)^2\right\}-\frac{1}{2\mu}\left(\frac{1}{r}+\frac{1}{s}\right)\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2\right\}$$

When the lens is of such a form that the angles of incidence and emèrgence are equal, the expressions are reduced to the following simple forms,

$$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+$$

$$+\frac{\theta^2 b^2}{2\mu f}\left\{(2\mu+1)\left(\frac{1}{s}+\frac{1}{b}\right)^2-\frac{1}{\mu-1}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2\right\}$$

$$\frac{1}{v_2}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2+\frac{\theta^2 b^2}{2\mu f}\left\{\left(\frac{1}{s}+\frac{1}{b}\right)^2-\frac{1}{\mu-1}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2\right\}$$

When the image is virtual, as in the case of a convex eye-lens and the eye at a distance from the lens, we have v_1 and v_2 negative, and the results may be investigated directly or may be obtained by putting $-\frac{1}{v_1}$ for $\frac{1}{v_1}$ and $-\frac{1}{v_2}$ for $\frac{1}{v_2}$ in the above.

CHAPTER IV.

ON THE FORMS OF THE IMAGES PRODUCED BY LENSES.

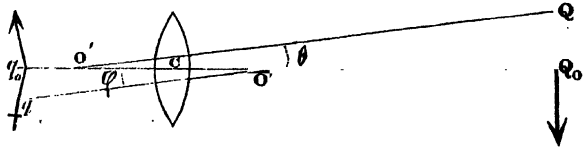
THE preceding Propositions give us the positions of the conjugate foci when the position of the luminous point, which is the origin of the incident pencil, is known, and when the position of the luminous origin is given in an algebraic equation which applies to all points in an object, we require a corresponding algebraic equation which shall in like manner apply to all points in the image, and which shall thus determine its form. In very many cases we have given the form of an image, formed by a mirror or lens, and have to determine the form of the secondary image as produced by a lens receiving the light from the first mirror or lens.

The focus which is conjugate to any luminous point in an object, must be generally taken *the cometa* or figure of oblique aberration, when formed by a centrical pencil; but must be taken *the least circle of confusion* if formed by an *excentrical* pencil, when the obliquity is *great* or the pencil *exceedingly small*. The first case is by far the most important in the construction of optical instruments, because the central parts of the field of view should be as accurately defined as possible; but when this is attained, it is frequently desirable to obtain also a large extent of *moderately accurate* field of view.

ART. 52. PROP. *To find the form of the image of* a straight line *placed directly before a convex lens.*

Taking the aberration as at Art. 45, page 113, and considering the *cometa* as in mirrors, page 33, to be situated at three-fourths

of the *least* longitudinal aberration from the first approximate



focus; also calling the distance $O'q = \rho$ we have

$$\rho = q - \frac{3}{4}\cdot\frac{q_1^2}{2\mu}\left\{\frac{\theta^2 dp'_1}{q'^3_1} + \frac{\mu-1}{\mu}\left[(\alpha-\theta)^2\left(\frac{p'_1}{q'_1}\right)^2\frac{\mu r + p_1}{r(p_1 - r)} + \right.\right.$$

$$\left.\left. + (\alpha' + \phi)^2\frac{\mu s + q_1}{s(q_1 - s)}\right]\right\}$$

Putting $p_1 = OQ_0$, $q_1 = O'q_0$, and p'_1, q'_1 the first approximate values of p', q' as before, we have to find $q = O'q$ for substitution, thus

$$\frac{1}{q} = \frac{\mu-1}{\mu s} + \frac{1}{\mu(p'+d)}, \qquad q'_1 = p'_1 + d$$

$$p' = \frac{rp}{(\mu-1)p - \mu r}, \text{ where } p = OQ,$$

then the lens being supposed very thin, we have

$$p = p_1 \text{ secant } \theta$$

$$= p_1\left(1 + \frac{\theta^2}{1.2}\right) \text{ nearly}$$

$$= p_1\left(1 + \frac{\phi^2}{2}\cdot\frac{q'^2_1}{p'^2_1}\right)$$

and after substituting, we have ρ expressed in the following form

$$\rho = q_1 - D\alpha^2 + E\alpha\phi - F\phi^2$$

Now unless E, three-fourths of the coefficient of $\alpha\phi$ in the

oblique aberration, vanishes for some particular case, the curve which is the image is not one of continued curvature at the axis of the lens, but is formed of two arcs of a spiral, meeting at a very obtuse angle, similar to the figure. When $E=0$, we have the form

$$\rho=q_1-D\alpha^2-F\phi^2$$

and comparing with the approximate polar equation of a circle, as at page 47, where we have the form

$$\rho=a'-\phi^2\cdot\frac{a'(a'-r')}{2r'}$$

and $$a'=q_1-D\alpha^2,$$

whilst r' is the radius of curvature required to be found, the curve being *concave* towards O'.

Equating the coefficients of ϕ^2, we have

$$r'=\frac{a'^2}{a'+2F}$$

$$=\frac{(q_1-D\alpha^2)^2}{q'_1-D\alpha^2+2F}$$

from which r' can be determined in any given case, by calculating D and F.

If the object be curved, we have only to find the approximate value of p in a similar manner, and can then calculate the radius of curvature of the image by the above formula.

Ex. I. To find the form of the image of the sun or moon given by a lens of the form determined in Ex. 1, page 118, when $E=0$.

We have now $$r=\frac{5}{9}\cdot f\,,\quad s=5f$$

$$p=\infty\,,\quad p'_1=2r\,,\quad q'_1=12r\,,\quad q_1=\frac{54}{5}r$$

$$\theta=6\phi\,,\quad \alpha'=\frac{\alpha}{9}\,,$$

These substituted in the expression for the aberration give us

the aberration $$=\frac{3}{5}r\alpha^2+\frac{243}{5}r\phi^2$$

and $$\rho=\frac{54}{5}r-\frac{3}{4}\left(\frac{3}{5}r\alpha^2+\frac{243}{5}r\phi^2\right)$$

and the radius of curvature $$r'=\frac{\left(\frac{54}{5}r-\frac{9}{20}r\alpha^2\right)^2}{\frac{54}{5}r-\frac{9}{20}r\alpha^2+\frac{3\times 243}{10}r}$$

Neglecting the term with α^2 as small, we have

$$r'=\frac{216}{155}r$$

$$=\frac{216}{155}\times\frac{5}{9}f$$

$$=\frac{24}{31}f$$

$$=\frac{3}{4}f \text{ nearly; }$$ see page 142.

Ex. 2. To find the curvature of the virtual image of a circular arc concentric with the first surface, and therefore nearly a straight line, given by the eye-lens determined in Ex. 4, page 119, when $r=45f$, $s=\frac{45}{89}\cdot f$

We have now $\quad r=45f, \quad s=\frac{45}{89}f$

$$p=p_1=\frac{51}{50}r, \qquad -p'_1=\frac{34}{33}r, \qquad -q'_1=\frac{r.56}{33\times 89}$$

$$-q_1=\frac{r.84}{5\times 89}$$

$$\theta=\phi\frac{28}{17\times 89}, \qquad \alpha'=89\alpha$$

which substituted in the expression for the aberration, gives the

aberration $=\alpha^2 r \cdot \frac{19278}{5}+\phi^2 r\,\frac{951426}{673285}$

and the image being virtual, with q_1 negative in respect of the standard case of Art. 45, if we call ρ the distance from O' to any point in the virtual image, we have

$$\rho=q_1+\frac{3}{4}\left(\alpha^2 r\,\frac{19278}{5}+\phi^2 r\,\frac{951426}{673285}\right)$$

which we have to compare with the approximate equation of a circle where the *convex* part is towards the pole, as follows,

$$\rho=a'+\phi^2\frac{a'(a'+r')}{2r'}$$

equating the coefficients of ϕ^2 and neglecting the term with α^2 as small compared with q_1 we find the radius r' of curvature of the image at the axis of the lens, thus,

$$r'=r\,\frac{5712}{309535}$$

$$=\frac{r}{54}\text{ nearly}$$

$$=\frac{5}{6}f$$

The lens being of short focus, the image is greatly curved, and only a small portion near the vertex can be considered free from distortion.

The effects of curvature in images are so important in the construction of optical instruments, that it is desirable to have our expressions put into a *general* form, for ready application to different cases, and for tracing the change of curvature in the image corresponding to a change of position in the object or primary image. When we have to consider the best forms of eye-pieces for telescopes and microscopes, we have to examine the effects of curvature in connexion with aberration and achromatism, a part of optics hitherto very imperfectly discussed by mathematicians.

ART. 53. PROP. *To investigate a general expression for the curvature of an image formed by a lens, when of continued curvature, the object or primary image being also of given continued curvature.*

Let r_1 be the radius of curvature of the object supposed to be *convex* towards O as pole, and p_1 being the distance of the point on the axis of the lens, let p be the distance of another point, we have as a circular arc, the approximate equation of the object, thus,

$$p=p_1+p_1\left(\frac{p_1}{2r_1}+\frac{1}{2}\right)\theta^2$$

where p_1 and r_1 will have the same sign when measured in the same direction from O, and different signs when r_1 is measured from the object towards O, or when it is *concave* towards O.

Using a similar notation to that of the last proposition,

let q_1 = the value of q on the axis, or when $\phi=0$,

$F=\frac{3}{4}$ the coefficient of ϕ^2 in the aberration,

F' = the coefficient of ϕ^2 in the term in q depending on the obliquity and the form of the object.

Then, ρ being the distance of a point in the image from O, we have the form

$$\rho=q\pm D\alpha^2\pm F\phi^2$$

where q is of the form

$$q=q_1+F'\phi^2$$

therefore, substituting and retaining only the upper sign for the more convenient use in comparing with the approximate equation of a circle and in tracing the direction in which the radius of curvature of the image is to be measured, we have the form

$$\rho=q_1+D\alpha^2+(F+F')\phi^2$$

and comparing with the approximate equation of a circular arc *convex* towards the pole, as follows,

$$\rho = a' + a'\left(\frac{a'}{2r'} + \frac{1}{2}\right)\phi^2$$

where r' is the radius of the circle, and a' the least distance of the circle from the pole, we have

$$a'\left(\frac{a'}{2r'} + \frac{1}{2}\right) = F + F'$$

and $a' = q_1$, if we neglect the small term $D\alpha^2$;

whence $$\frac{1}{r'} = \frac{2F}{q_1^2} + \frac{2F'}{q_1^2} - \frac{1}{q_1}$$

Now in the quantity $\frac{a'}{2r'}$ we have a' and r' with the *same* sign when the arc is *convex* to the pole, but of different signs when it is *concave* to the pole. Therefore if in any particular case, we find r' to be of the same sign as q_1, the image is *convex* towards O' as pole, but *concave* when we find r' and q_1 to be of different signs.

Referring to Art. 52, and remembering that $\theta = \phi \cdot \frac{q'_1}{p'_1}$ we have

$$F = -\frac{3}{8}\frac{q_1^2}{\mu}\left\{\frac{d}{p'_1 q'_1} + \frac{\mu - 1}{\mu}\left(\frac{\mu r + p_1}{r(p_1 - r)} + \frac{\mu s + q_1}{s(q_1 - s)}\right)\right\}$$

and since the term in the aberration with $\phi\alpha$ vanishes, when the image is of continued curvature, we have from Art. 46,

$$\frac{r}{s} \cdot \frac{\mu s + q_1}{s(q_1 - s)} - \frac{p'_1}{q'_1} \cdot \frac{\mu r + p_1}{r(p_1 - r)} = 0$$

or $$\mu^2 - m(2\mu^2 - \mu - 1) - n(\mu^2 - 1) = 0$$

this condition being established, and proceeding in the same manner as in that article, we find

$$F = -\frac{3}{8} \cdot \frac{q_1^2}{\mu}\left\{\frac{\mu^2(\mu - 1)(1 - 2m)^2}{f[\mu - m(\mu - 1)][1 + m(\mu - 1)]} + \frac{\mu}{f}\right\}$$

Again comparing the expression

$$q = q_1 + F'\phi^2$$

with the value of q determined by means of the following

$$\frac{1}{q}=\frac{\mu-1}{\mu s}+\frac{1}{\mu(p'+d)}$$

$$p'=\frac{rp}{(\mu-1)p-\mu r}$$

$$p=p_1+p_1\left(\frac{p_1}{2r_1}+\frac{1}{2}\right)\theta^2$$

we find
$$F'=-\frac{q_1^2}{p_1}\left(\frac{p_1}{2r_1}+\frac{1}{2}\right)$$

$$=-\frac{q_1^2}{2}\left(\frac{1}{r_1}+\frac{1}{p_1}\right)$$

Now r_1 must be given, therefore let it $=bf$, also expressing as before p_1 in terms of f, m and μ, we have

$$\frac{2F'}{q_1^2}=-\frac{1}{f}\left(\frac{1}{b}+\frac{m[\mu^2-m(2\mu^2-\mu-1)]}{\mu[\mu-m(\mu-1)]}\right)$$

also q_1 being expressed in the same way gives

$$\frac{1}{q_1}=\frac{[m(2\mu^2-\mu-1)-\mu^2+\mu+1](1-m)}{f\mu[1+m(\mu-1)]}$$

These expressions enable us to calculate the magnitude and direction of r' for given values of f, m and μ; but it will be convenient to take the ordinary value of μ, or $\mu=\frac{2}{3}$, and simplify the expression still further; in this way we find

$$\frac{1}{r'}=\frac{2F}{q_1^2}+\frac{2F'}{q_1^2}-\frac{1}{q_1}$$

$$=-\frac{3}{4f}\left(\frac{9-11m+11m^2}{(3-m)(2+m)}\right)-\frac{1}{f}\left(\frac{1}{b}+\frac{m(9-8m)}{3(3-m)}\right)-\frac{(8m+1)(1-m)}{f3(2+m)}$$

from which the value of r' is easily found for given values of m and b.

Ex. 1. To apply the formula to finding the radius of curvature of the image of the sun or moon given by a convex lens, as in Ex. 1, page 137.

We have now $u=\infty \quad \therefore m=\frac{f}{u}=0$

$$p=p_1+p_1\left(\frac{p_1}{2r_1}+\frac{1}{2}\right)\theta^2=p_1 \text{ nearly, and } F' \text{ vanishes};$$

$$\therefore \frac{1}{r_1}=-\frac{1}{f}\left(\frac{9}{8}+\frac{1}{6}\right)$$

$$=-\frac{31}{24f}$$

or

$$r'=-\frac{24}{31}f$$

$$=-\frac{3}{4}f \text{ nearly, as found before by the}$$

other method, page 137, and the image is *concave* towards the lens, since q_1 is positive and r' is negative.

Ex. 2. When an object which is a straight line, has its *virtual* image given by an eye-lens, magnified 10 times, required the curvature of the image.

Then
$$\frac{u}{v}=\frac{1}{10}=u\left(\frac{1}{u}-\frac{1}{f}\right)$$

$$=1-\frac{1}{m}$$

$$\therefore \quad m=\frac{10}{9} \qquad \text{also} \quad r_1=\infty=bf, \ \therefore \ \frac{1}{b}=0$$

and substituting this value of m in the formula for $\frac{1}{r'}$ we have

$$\frac{1}{r'}=-\frac{63027}{51408 . f}$$

and
$$r'=-\frac{5}{6}f \text{ nearly, as in Example 2, page 138,}$$

also q_1 is now negative, therefore the image is *convex* to O' and to the lens.

Ex. 3. To find the curvature of the image when the lens acts as a field-glass in the eye-tube of a microscope or telescope.

The pencils converging from the object-glass to form the primary image, are now intercepted by the field-lens and we have m negative, of different values in different cases;

As a simple case, let $m=\dfrac{f}{-u}=-1$

we have $$\frac{1}{r'}=\frac{1}{f}\left\{\frac{13}{48}-\frac{1}{b}\right\}$$

if $b=24$, or $r_1=24f$ for the case of a telescope

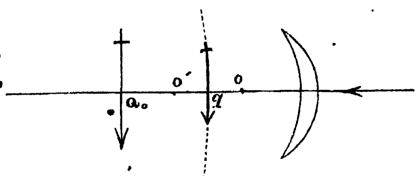
then $$\frac{1}{r'}=\frac{1}{f}\cdot\frac{11}{48}$$

and $$r'=\frac{48}{11}f=4f \text{ nearly}$$

and $q_1=-\dfrac{3}{14}f$ is negative, therefore the image is concave to O'.

We find for this case

$r=\dfrac{5}{17}f$, $s=-\dfrac{5}{7}f$, $v=\dfrac{f}{2}$,



and the circumstances, are those of the figure.

In the case of microscopes b will be always smaller, as seen in the next Example, but if not less than 4, the image will be still *convex* to the field-glass, and *concave* to the eye-glass, thus allowing a larger portion of the image to be seen distinctly than would have been if the field-glass had not been interposed and the original image *convex* to the eye-glass had been viewed; and hence the name of field-glass from this property of increasing the extent of the distinct part of the field of view.

Ex. 4. To find the curvature of the image of a straight line formed a single convex lens, as the power of a compound microscope.

Let $v=10u$, $\therefore\ \dfrac{1}{v}=\dfrac{1}{10u}=\dfrac{1}{f}-\dfrac{1}{u}$

$$\frac{10}{11}u=f \qquad\qquad m=\frac{10}{11}$$

applying this value in the formula for r' we have

$$\frac{1}{r'}=-\frac{11783}{8832f}$$

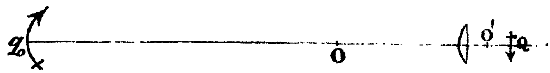
or
$$r'=-\frac{3}{4}f \quad \text{nearly}$$

and q_1 is positive, therefore the image is concave towards O' and the lens.

We have now
$$n=\frac{9-8m}{5}=\frac{19}{55}$$

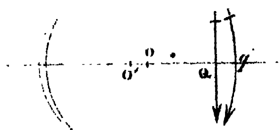
$$\therefore\ r=\frac{55}{19}f \qquad \text{and } s=\frac{f}{2-n}=\frac{55}{91}f$$

and the circumstances are as in the figure.



Ex. 5. To find the curvature of the image when a primary image is *near* the field-glass of an eye-piece, similar to Ramsden's.

We must now suppose u very small compared with the focal length of the lens and $u=\frac{f}{m}$ $\therefore$ m is very large, and we need only take the highest powers of m in the expression for $\frac{1}{r'}$, then we find $\frac{1}{r'}=\frac{33}{4f}-\frac{1}{fb}$.



If $b=\infty$ for a straight line, then $r'=\frac{4}{33}f$ and q_1 is negative, and r' and q_1 being of opposite signs, the image is concave to O' and the lens; the value of $r=-\frac{5}{8}u$ is very small compared with f, and s positive is very little smaller, therefore the meniscus is

nearly a spherical shell and the circumstances like the figure. When we have discussed the achromatism of eye-pieces, we shall see that if we use such a lens as this in order to procure a large field of view, we shall not be able to satisfy the condition for achromatism.

ART. 54. *When an image is considered the locus of the circle of confusion, for very small pencils or considerable obliquities, we can find the form of the image by calculating a succession of points on the respective refracted rays at the distances,* $\frac{v_1+v_2}{2}$ *from the lens.*

We cannot correctly, however, consider an image as formed in this manner near the center of the field of view, where the obliquity is very small, and where the aberration in even a small pencil becomes very large compared with the confusion. The radius of curvature of an image, at the axis, as if formed by indefinitely small pencils, will, however, be found in the next Article.

From Articles 42, page 103, and 49, page 125, we have to use the expressions for *very thin* lenses as follows:

$$\frac{1}{v_1}=\left(\frac{\mu\ \cos.\ i'}{\cos.\ i}-1\right)\left(\frac{1}{r}+\frac{1}{s}\right)\sec.\ i-\frac{1}{u}$$

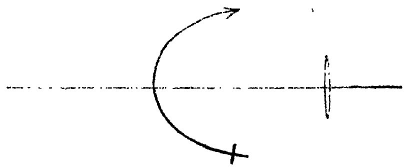
$$\frac{1}{v_2}=\left(\frac{\mu\ \cos.\ i'}{\cos.\ i}-1\right)\left(\frac{1}{r}+\frac{1}{s}\right)\cos.\ i-\frac{1}{u}$$

If we take $\mu=\frac{3}{2}$, the lens equiconvex, and the object directly before the lens, but so distant that we may neglect $\frac{1}{u}$, when the focal length $=1.=\frac{v_1+v_2}{2}$ for $i=0$, we have the series of values for $\frac{v_1+v_2}{2}$ as follow:

$$\text{for } i = 0, \quad \frac{v_1 + v_2}{2} = 1.$$
$$i = 10^0, \quad \ldots\ldots = .975$$
$$i = 20^0, \quad \ldots\ldots = .904$$
$$i = 30^0, \quad \ldots\ldots = .798$$
$$i = 40^0, \quad \ldots\ldots = .673$$
$$i = 45^0, \quad \ldots\ldots = .609$$

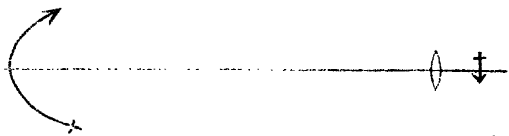
which give an image as in the figure, drawn to a focal length of one inch.

This form of the image with the attendant distortion can be easily seen with a small lens of one inch focus, by receiving the image which it gives of the bars of a window, on a piece of paper bent to the required curvature.



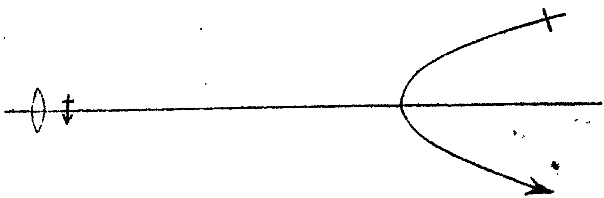
The distortion or change in the proportions of different corresponding parts of the image and object depends chiefly on the change in the relative distances of these corresponding parts from the lens.

If we examine the form of a real magnified image of a straight line,



we find it as in the figure, and the distortion, except near the axis, is very great, the figure being drawn for a focal length of ¼ inch.

In the case of a virtual magnified image we find, as in the



annexed figure, to a focal length of $\frac{1}{4}$ inch, and the distortion is now again very great at the boundaries of the field of view, these parts being now extended, whilst in the former cases they were compressed.

Art. 55. Prop. *To find the radius of curvature of an image of a given object at the axis, when considered the locus of the circle of confusion.*

We have now the obliquity i very small, and considering the lens very thin, we must use the expressions of Art. 43, page 108, in the following form,

$$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u}+\frac{i^2(2\mu+1)}{2\mu f}$$

$$\frac{1}{v_2}=\frac{1}{f}-\frac{1}{u}+\frac{i^2}{2\mu f}$$

where u is the distance of any point in the object from the lens; let u_0, v_0 be the values on the axis when v_1 and v_2 each $=v_0$.

If r_1 be the radius of curvature of the object, we shall have as in the previous articles,

$$u=u_0+\frac{i^2 . u_0}{2}\left(\frac{u_0}{r_1}+1\right)$$

$$\therefore \quad \frac{1}{u}=\frac{1}{u_0}-\frac{i^2}{2}\left(\frac{1}{r_1}+\frac{1}{u_0}\right)$$

and substituting

$$\frac{1}{v_1}=\frac{1}{f}-\frac{1}{u_0}+\frac{i^2}{2}\left(\frac{1}{r_1}+\frac{1}{u_0}+\frac{2\mu+1}{\mu f}\right)$$

$$\frac{1}{v_2}=\frac{1}{f}-\frac{1}{u_0}+\frac{i^2}{2}\left(\frac{1}{r_1}+\frac{1}{u_0}+\frac{1}{\mu f}\right)$$

and since $$\frac{1}{v_0}=\frac{1}{f}-\frac{1}{u_0}$$

we have $$v_1=v_0-\frac{i^2 . v_0^2}{2}\left(\frac{1}{r_1}+\frac{1}{u_0}+\frac{2\mu+1}{\mu f}\right)$$

$$v_2 = v_0 - \frac{i^2 \cdot v_0^2}{2}\left(\frac{1}{r_1} + \frac{1}{u_0} + \frac{1}{\mu f}\right).$$

Putting ρ = the distance of the circle of confusion from the lens, we have

$$\rho = \frac{v_1 + v_2}{2} = v_0 - \frac{i^2 \cdot v_0^2}{2}\left(\frac{1}{r_1} + \frac{1}{u_0} + \frac{\mu + 1}{\mu f}\right)$$

and comparing this with the approximate equation of a circle, whose radius $= r'$, and being supposed *convex* to the pole, as before, has its least distance $= v_0$, as follows,

$$\rho = v_0 + \frac{i^2 \cdot v_0^2}{2}\left(\frac{1}{r'} + \frac{1}{v_0}\right)$$

$$\therefore \quad \frac{1}{r'} + \frac{1}{v_0} = \frac{1}{r'} + \frac{1}{f} - \frac{1}{u_0}$$

$$= -\left(\frac{1}{r_1} + \frac{1}{u_0} + \frac{\mu + 1}{\mu f}\right)$$

or
$$\frac{1}{r'} = -\frac{1}{r_1} - \frac{2\mu + 1}{\mu f}$$

or, the curvature of the image depends on the curvature of the object and the focal length of the lens, but is *independent of the distance* of the object from the lens.

Now $\rho = \frac{v_1 + v_2}{2}$ being positive, and r' negative, whilst r_1 is positive for an object *convex* towards the lens, equal infinity for a *plane* object, or for a *concave* object r_1 negative but $\frac{1}{r_1} < \frac{2\mu + 1}{\mu f}$, we have the image *concave* towards the lens.

When $-\frac{1}{r_1} = \frac{2\mu + 1}{\mu f}$ the image will be plane, or the object may be so *concave* towards the lens as to give a flat image, and any greater concavity than this will give an image convex to the lens.

Again, when the image is virtual we have $\rho = \frac{v_1 + v_2}{2}$ negative,

and of the same sign as r' whilst the object is plane, convex towards the lens, or only *slightly* concave; therefore this virtual image is *convex* to the lens.

This discussion shews us that in such instruments as the magic-lantern or oxy-hydrogen microscope, the pictures should be painted on a curved surface with radius r_1 such that $\frac{1}{r_1} = -\frac{2\mu+1}{\mu f}$, and then r' will be infinite or the image will be plane, and may be received on a flat screen. Taking $\mu = \frac{3}{2}$ the above expression gives $r_1 = \frac{3}{8}f$. When the object is plane or $r_1 =$ infinity, then the radius of curvature of the image $= r' = \frac{3}{8}f$.

CHAPTER V.

ON ACHROMATIC AND APLANATIC COMBINATIONS.

IN the Chapter on Chromatics, PART I., it was stated that the *dispersive power* of a medium $= \frac{\delta\mu}{\mu - 1}$ was best determined from measures of the refractive indices of Fraunhofer's fixed lines by taking $\delta\mu =$ the difference of μ, for the fixed lines D and F, and the μ in the denominator the mean between them. For practical purposes, another method of determining *the ratio* of the dispersive powers of two substances would be more frequently employed; and this dispersive ratio is all that is required in forming *double* achromatic prisms or lenses: that is, we do not require to know p_1 or p_2 the separate dispersive powers, but only their ratio, say

$$\varpi = \frac{p_1}{p_2}.$$

For this practical method of determining ϖ, without the laborious and delicate experiments with the Fraunhofer spectrum, see page 157, Art 58.

When we have *triple* combinations of either prisms or lenses, as in Art. 86, PART I., we may combine three of the fixed lines, and thus reduce the residual spectrum which arises from *the irrationality* of the dispersion. This requires another method of treating the dispersion, and Sir John Herschel has proposed to refer the dispersions of different media to that of water at a fixed temperature as a standard, in the following manner. (See the article 'Light' in the *Encyclopædia Metropolitana.*)

Let μ_0 be the refractive index for the given substance and the fixed line B; x_0 the refractive index for water and the same fixed

line. Let μ and x be the refractive indices respectively for some other fixed line, and assume the following series

$$\frac{\delta\mu}{\mu_0-1}=a\frac{\delta x}{x_0-1}+b\left(\frac{\delta x}{x_0-1}\right)^2+c\left(\frac{\delta x}{x_0-1}\right)^3+\&c.$$

Now if E be taken for the other fixed line, and we find $\delta\mu$ and δx in numbers, we have the constant a given by the numerical equation

$$\frac{\delta\mu}{\mu_0-1}=a\frac{\delta x}{x_0-1}$$

If we find $\delta\mu$ and δx for B and another fixed line as H in the violet, we shall have two equations of the form

$$\frac{\delta\mu}{\mu_0-1}=a\frac{\delta x}{x_0-1}+b\left(\frac{\delta x}{x_0-1}\right)^2$$

which will give values to both a and b, and so we may proceed onwards for any number of fixed lines.

Sir John Herschel has given a table of the coefficients a and b for the substances contained in Fraunhofer's table of refractive indices at the end of PART I.

The series which expresses the equivalent of $\frac{\delta\mu}{\mu_0-1}$ cannot be given in a general form, but must be adapted to each particular case; thus, when we have a *double* achromatic lens or prism to discuss, we should use the lines D and F, or perhaps C and F, to determine the value of a; but if we had a *triple* combination to discuss, we should probably choose the lines C, E and G to determine a and b.

The angle in which the spectrum is dispersed changes with the incidence on the prism, and a prism of a smaller angle at one incidence will give an angular dispersion as great as a prism of larger angle at another incidence. By placing two prisms made from the same material, but of different angles, with their edges in opposite directions, a combination nearly achromatic may be formed, which gives considerable deviation to the pencil.

An object viewed through such a combination, appears extended

in one direction but without colour; and if another equal combination be set with the principal section of the prisms at right angles to that of the first, the object appears extended in the direction at right angles to the first, or it appears magnified. This constitutes the prismatic telescope of Professor Amici, of Modena, but the combination had been long previously studied in Scotland by Dr. Blair and Sir David Brewster.

Prisms placed in any other position than that for the minimum deviation, we saw in Art. 29, were affected with *confusion*, so that to produce distinctness, the pencil should pass through each of the prisms in an achromatic combination at the angle of minimum deviation, when the angles of the prisms are not small.

ART. 56. PROP. *To investigate an expression for finding the forms of two prisms in an achromatic combination, when the pencils pass through each with the minimum deviation.*

Let d_1 and d_2 be the deviations produced by the prisms whose angles are α_1 and α_2, and refractive indices μ_1 and μ_2 respectively, and d_1+d_2 the deviation produced in the pencil by both.

When there is achromatism, the differential of (d_1+d_2) taken with respect to the variation of the refractive indices, must $=0$.

or $$d(d_1)+d(d_2)=0$$

but at the angle of minimum deviation, PART I., Art. 35, we

have $$\mu_1 \sin. \frac{\alpha_1}{2} = \sin.\left(\frac{d_1+\alpha_1}{2}\right)$$

$$\therefore \quad \sin. \frac{\alpha_1}{2} \cdot d\mu_1 = \frac{1}{2}\cos. \frac{d_1+\alpha_1}{2} \cdot d(d_1)$$

$$= \frac{1}{2}\sqrt{1-\mu_1^2 \sin.^2 \frac{\alpha_1}{2}} \cdot d(d_1)$$

and similarly for $d(d_2)$

$$\therefore \quad 0 = \frac{\sin. \frac{\alpha_1}{2} \cdot d\mu_1}{\sqrt{1-\mu_1^2 \sin.^2 \frac{\alpha_1}{2}}} + \frac{\sin. \frac{\alpha_2}{2} \cdot d\mu_2}{\sqrt{1-\mu_2^2 \sin.^2 \frac{\alpha_2}{2}}}$$

or putting $$p_1=\frac{d\mu_1}{\mu_1-1}, \quad p_2=\frac{d\mu_2}{\mu_2-1}$$

we have $$\frac{p_1}{p_2}=-\frac{\sin.\frac{\alpha_2}{2}\sqrt{1-\mu_1^2\sin.^2\frac{\alpha_1}{2}}\cdot(\mu_2-1)}{\sin.\frac{\alpha_1}{2}\sqrt{1-\mu_2^2\sin.^2\frac{\alpha_2}{2}}\ (\mu_1-1)}$$

which gives the relation between the dispersive powers and angles of the prisms required; the negative sign shewing that the refracting angles of the prisms must be turned in opposite directions.

ART. 57. PROP. *To find the conditions of the most complete achromatism to be obtained by a triple lens.*

First, to find the differential of $\frac{1}{f}$ in respect of the variation of μ, we have

$$\frac{1}{f}=(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)$$

$$d\left(\frac{1}{f}\right)=d\mu\left(\frac{1}{r}+\frac{1}{s}\right)$$

$$=p(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)$$

$$=\frac{p}{f}$$

From Art. 80, PART I., we have for a combination of lenses when the aberration is neglected,

$$\frac{1}{v}=\Sigma\left(\frac{1}{f}\right)-\frac{1}{u}$$

or, for a triple lens when the focal lengths are f_1, f_2, f_3, and dispersive powers p_1, p_2, p_3, respectively

$$\frac{1}{v}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}-\frac{1}{u}$$

and the differential of v must $=0$ for the variations of the refractive indices

$$\therefore \quad 0=\frac{p_1}{f_1}+\frac{p_2}{f_2}+\frac{p_3}{f_3}$$

or using the formula of Sir John Herschel, we have

$$0=\frac{1}{f_1}\left\{a_1\frac{\delta x}{x_0-1}+b_1\left(\frac{\delta x}{x_0-1}\right)^2+\&c.\right\}$$
$$+\frac{1}{f_2}\left\{a_2\frac{\delta x}{x_0-1}+b_2\left(\frac{\delta x}{x_0-1}\right)^2+\&c.\right\}$$
$$+\frac{1}{f_3}\left\{a_3\frac{\delta x}{x_0-1}+b_3\left(\frac{\delta x}{x_0-1}\right)^2+\&c.\right\}$$

where the coefficient of $\frac{1}{f_1}$ is the value of p_1, and those of $\frac{1}{f_2}$, $\frac{1}{f_3}$, the values of p_2 and p_3 respectively.

Now in order that the equation may be satisfied independently of any particular value of $\frac{\delta x}{x_0-1}$, we must have the coefficients of each power separately $=0$, or

$$0=\frac{a_1}{f_1}+\frac{a_2}{f_2}+\frac{a_3}{f_3}$$

$$0=\frac{b_1}{f_1}+\frac{b_2}{f_2}+\frac{b_3}{f_3}$$

Let the focal length of the combination $=F$, or

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}$$

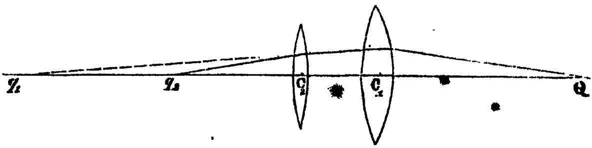
These three equations suffice to determine f_1, f_2 and f_3, of which one at least must be negative, and another positive.

In the triple lens we are thus able to unite the three of the fixed lines of the spectrum which were employed in calculating the coefficients a_1, a_2, a_3, b_1, b_2, b_3, and thus reduce greatly the residual spectrum.

Art. 58. Prop. *To find the condition of achromatism in two lenses separated by a given interval.*

Let the lenses, in the first instance, be supposed as in the figure;

and f_1 the focal length of c_1, f_2 that of c_2, $v_1 = c_1q_1$, $v_2 = c_2q_2$,



$Qc_1 = u$, and the distance $c_1\,c_2$ between the lenses $= a$,

then
$$\frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u}$$
$$\frac{1}{v_2} = \frac{1}{f_2} + \frac{1}{v_1 - a}$$

Now, when q_2 is an achromatic focus, the differential of v_2 in respect of a variation of μ, must $= 0$; therefore, differentiating the above equations, we have

$$-\frac{dv_1}{v_1^2} = \frac{p_1}{f_1}$$
$$-\frac{dv_2}{v_2^2} = \frac{p_2}{f_2} - \frac{dv_1}{(v_1 - a)^2}$$

or
$$0 = \frac{p_2}{f_2} - \frac{dv_1}{v_1^2} \cdot \frac{1}{\left(1 - \frac{a}{v_1}\right)^2}$$
$$= \frac{p_2}{f_2} + \frac{p_1}{f_1} \frac{1}{\left\{1 - a\left(\frac{1}{f_1} - \frac{1}{u}\right)\right\}^2}$$

and
$$\frac{p_1}{p_2} = -\frac{f_1}{f_2}\left\{1 - a\left(\frac{1}{f_1} - \frac{1}{u}\right)\right\}^2$$

which gives the relation required, the negative sign shewing that one of the lenses must be concave, and the other convex.

When a is small we may neglect a^2 and have

$$\frac{p_1}{p_2} = -\frac{f_1}{f_2}\left\{1 - 2a\left(\frac{1}{f_1} - \frac{1}{u}\right)\right\}.$$

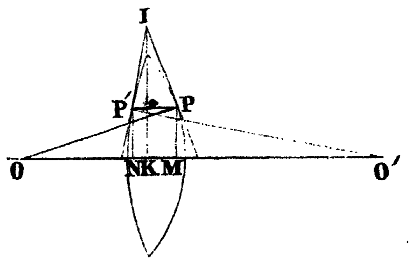
This discussion shews us that when the achromatism of two

lenses in contact is not complete, it may be improved by separating them; but the condition is not then independent of u.

This formula enables us to determine the dispersive ratio $\varpi = \frac{p_1}{p_2}$, when we have found the focal lengths of two lenses which form a combination *nearly* achromatic, by measuring the distance of separation a which renders the achromatism complete. This is the method referred to at the beginning of this Chapter, and is independent of any hypothesis as to what rays should be taken in order to secure the best result in practice.

ART. 59. The results of the previous Propositions, which are treated on the usual method in achromatism, must be considered only first approximations. We may consider a lens as formed of an infinite number of virtual prisms, whose faces are the tangent planes to the surfaces of the lens. Now if a ray always passed through the lens parallel to the axis, our ordinary expressions are easily obtained by the properties of prisms with small refracting angles, but as the rays pass obliquely in general through the lenses, there is strictly a small correction remaining, which depends on the difference of the distances from the axis at which the ray is incident and emergent, that is upon the thickness of the lens and the inclination of the ray to the axis. In telescopic object-glasses this correction is always small.

To shew how the usual expression for the achromatism of a double object-lens may be obtained by the consideration of virtual prisms; let O be the center of curvature of the first, O' that of the second surface, PP' a line parallel to the axis OO'; PI, PI' tangents at P and P'; then PIP' is the



refracting angle of the virtual prism through which a ray PP' passes. Draw the perpendiculars $PM = P'N = y$ say, and IK, then the tangents being perpendicular to the radii, we have

$$\begin{aligned}\angle I &= \angle PIK + \angle P'IK \\ &= \angle POM + \angle P'O'N \\ &= \frac{y}{r} + \frac{y}{s}\end{aligned}$$

and the deviation between the incident and emergent rays

$$\begin{aligned}= D &= (\mu - 1) I \\ &= (\mu - 1) y \left(\frac{1}{r} + \frac{1}{s}\right) \\ &= \frac{y}{f}\end{aligned}$$

similarly in the concave lens we should have for the deviation

$$D' = -\frac{y}{f'}$$

and
$$D + D' = y\left(\frac{1}{f} - \frac{1}{f'}\right)$$

and differentiating in respect of μ and μ', we have

$$\begin{aligned}d(D + D') &= 0 \text{ for achromatism} \\ &= y\left(\frac{p}{f} - \frac{p'}{f'}\right)\end{aligned}$$

or
$$0 = \frac{p}{f} - \frac{p'}{f'}$$

as found in Art. 85, PART I.

If the ray passes obliquely through the lens, the above result is not quite true. The student who feels interested in the subject will find the problem discussed for a telescopic lens, in a paper by the author, in the Transactions of the Cambridge Philosophical Society, Vol. VI. Part III., but the process is too long for insertion in the present treatise. The following is the equation for achromatism,

$$\begin{aligned}0 = \frac{\delta\mu}{\rho_1} - \frac{\delta\mu'}{\rho_2} &+ t_1\left\{\left(\frac{\mu - 1}{\mu} \cdot \delta\mu' + \frac{\mu' - 1}{\mu^2} \cdot \delta\mu\right)\frac{1}{r_1 \rho_2} - \frac{\mu^2 - 1}{\mu^3} \cdot \frac{\delta\mu}{r_1 r_2}\right\} \\ &+ t_2\left\{\left(\frac{\mu' - 1}{\mu'} \cdot \delta\mu + \frac{\mu - 1}{\mu'^2} \cdot \delta\mu'\right)\frac{1}{r_3 \rho_1} - \frac{\mu'^2 - 1}{\mu'^2} \cdot \frac{\delta\mu'}{r_2 r_3}\right\}\end{aligned}$$

where t_1 is the thickness of the convex lens at its edge, r_1, r_2, the radii of the first and second surfaces; t_2, r_2, r_3 the same quantities for the concave lens, which has the same curvature at its inner surface as the convex; also μ the refractive index for the convex, μ' for the concave, and

$$\frac{1}{\rho_1}=\left(\frac{1}{r_1}+\frac{1}{r_2}\right),\ \frac{1}{\rho_2}=\left(\frac{1}{r_2}+\frac{1}{r_3}\right)$$

The investigation would require modification for the object-lenses of microscopes on account of the proximity of the object to the lens, with the larger proportion of the aperture and t_1, t_2, to the radii of the surfaces. The correction would probably rise to a larger amount than in telescopic lenses, but in practice these corrections would appear to the working optician as defective achromatism, and would be easily remedied by changing the curvature of one of the surfaces.

The achromatism of eye-pieces was explained in Art. 87, Part I. to be produced by the lenses being placed at such distance that the coloured rays, which originally constituted a ray of white light, emerged from the eye-lens in a state of parallelism; the result being produced by the ray which had the most deviation at the first lens, having the least at the second one.

A similar effect takes place in a single lens, at the two surfaces, when the thickness is considerable.

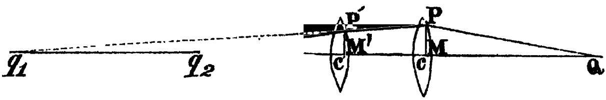
Euler, to whom we are indebted for the method of investigation used in the next article, has discussed in his work "Dioptrica," the case of a single thick lens, as well as the cases of several lenses. His method consists in taking the angles which the differently coloured rays make with the axis of the lens at emergence to be the same; when they have originally constituted one compound ray. Thus if $\phi = \angle P'q_2M'$ in the next figure, we must have

$$\frac{d\phi}{d\mu}=0$$

and therefore $$\frac{d.\tan.\phi}{d\mu}=0.$$

ART. 60. PROP. *To investigate the condition that two lenses of the same kind of glass may form an achromatic eye-piece.*

Let f_1 be the focal length of the lens C, f_2 that of C', and the



distance $CC'=a$; $QPP'q_2$ being the course of a ray.

Let $QC=u$, $q_1C=v$, $q_2C'=v'$, $PM=y$, $P'M'=y'$;

then
$$\frac{1}{v}=\frac{1}{f_1}-\frac{1}{u}$$

$$\frac{1}{v'}=\frac{1}{f_2}+\frac{1}{v-a}$$

$$\frac{y'}{y}=\frac{q_1M'}{q_1M}=\frac{v-a}{v}$$

and, tan. $P'q_2M'$ is to be the same for rays of all colours, or $d(\tan.\ P'q_2M')=0$

but,
$$\tan.\ P'q_2M'=\frac{y'}{v'}$$

$$=\frac{y(v-a)}{v}\left\{\frac{1}{f_2}+\frac{1}{v-a}\right\}$$

$$=y\left\{\frac{1}{f_2}\left(1-\frac{a}{v}\right)+\frac{1}{v}\right\}$$

$$=y\left\{\frac{1}{f_1}+\frac{1}{f_2}-a\left(\frac{1}{f_1f_2}-\frac{1}{f_2u}\right)-\frac{1}{u}\right\}$$

differentiating with respect to μ, and equating to zero, we have, since y is the same for each coloured ray in QP,

$$0=\frac{p}{f_1}+\frac{p}{f_2}-a\frac{1}{f_2}\cdot\frac{p}{f_1}-\frac{a}{f_1}\cdot\frac{p}{f_2}+\frac{a}{u}\cdot\frac{p}{f_2}$$

since p the dispersive power is the same for both lenses, and

$$0=\frac{1}{f_1}+\frac{1}{f_2}-a\left(\frac{2}{f_1f_2}-\frac{1}{uf_2}\right)$$

$$\therefore \quad a = \frac{f_1 + f_2}{2 - \frac{f_1}{u}}$$

which gives the distance a required.

For the eye-piece of a telescope u is the distance of the field-glass from Q the center of the object-glass, and being large compared with f_1 we may consider $\frac{1}{u} = 0$, which makes the investigation more simple for that case, and gives

$$a = \frac{f_1 + f_2}{2}$$

For the achromatic eye-piece of a microscope $\frac{f_1}{u}$ cannot be neglected, and we see that the lenses should be more separated than for a telescope.

By the same method the condition of achromatism in an eye-piece consisting of three or four lenses may be investigated; but in the case of three lenses one of the distances is arbitrary, and in the case of four lenses two of the distances are arbitrary. The student will find both cases investigated in Mr. Coddington's Treatise, Articles 209 to 212.

Aplanatism, or freedom from aberration, has been already discussed for single lenses in Art. 87; we have now to consider it for combinations of lenses, where the aberration of one of the component lenses is destroyed by the contrary aberration of the other, or others.

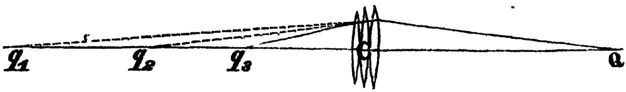
The small term in the value of $\frac{1}{v}$ being its variation, depending on xr (or $y^2 = 2xr$ nearly), and the form of the lens, together with the refractive index, we may write it $\delta \cdot \frac{1}{v}$, or

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u} + \delta \cdot \frac{1}{v}$$

and we may use $\delta \frac{1}{v}$ to either a *second* or *third* approximation according to the requirements of the particular case under discussion.

ART. 61. PROP. *To investigate the general condition of aplanatism in a series of thin lenses in contact.*

Let $QC=u$, f_1, f_2, f_3, &c., the focal lengths of the lenses in succession from Q,



$$v_1=q_1C,\ v_2=q_2C,\ v_3=q_3C,\ \&c.$$

then we have, as above,

$$\frac{1}{v_1}=\frac{1}{f_1}-\frac{1}{u}+\delta\cdot\frac{1}{v_1}$$

$$\frac{1}{v_2}=\frac{1}{f_2}+\frac{1}{v_1}+\delta\cdot\frac{1}{v_2}$$

$$\frac{1}{v_3}=\frac{1}{f_3}+\frac{1}{v_2}+\delta\cdot\frac{1}{v_3}$$

&c. &c.

or, performing the summation we have, if we put n for the number of the lenses,

$$\frac{1}{v_n}=\Sigma\left(\frac{1}{f}\right)-\frac{1}{u}+\Sigma\left(\delta\cdot\frac{1}{v}\right)$$

and the aberration $=\delta\,.\,v_n=-v_n^2\Sigma\left(\delta\,.\,\frac{1}{v}\right)$

Now in order that the combination may be aplanatic, we must have

$$\delta\,.\,v_n=0$$

or

$$\Sigma\left(\delta\cdot\frac{1}{v}\right)=0$$

It was found in Art. 35 that in certain cases, convex lenses of the meniscus form had opposite aberration to what is the general direction, and hence combinations of a meniscus with another convex lens may be aplanatic by the opposite aberrations destroying each other. Aplanatic lenses, however, are much better formed according to the method of the next article, so that achromatism is secured at the same time as aplanatism.

Cor. When the lenses are separated by given intervals a_1, a_2, a_3, &c., the expressions become as follow,

$$\frac{1}{r_1}=\frac{1}{f_1}-\frac{1}{u}+\delta\left(\frac{1}{r_1}\right)$$

$$\frac{1}{r_2}=\frac{1}{f_2}+\frac{1}{r_1-a_1}+\delta\left(\frac{1}{r_2}\right)$$

$$\frac{1}{r_3}=\frac{1}{f_3}+\frac{1}{r_2-a_2}+\delta\left(\frac{1}{r_3}\right)$$

&c. &c.

$$\frac{1}{r_n}=\frac{1}{f_n}+\frac{1}{r_{n-1}-a_{n-1}}+\delta\left(\frac{1}{r_n}\right)$$

and $$\therefore \frac{1}{r_n}=\Sigma\left(\frac{1}{f}\right)+\Sigma\left\{\frac{1}{r-a}-\frac{1}{r}\right\}+\Sigma\left\{\delta\left(\frac{1}{r}\right)\right\}-\frac{1}{u},$$

and the sum $\Sigma\left\{\delta\left(\frac{1}{r}\right)\right\}$ is to be formed, by the numerical calculation of the value of each of its terms $\delta\left(\frac{1}{r_1}\right)$, $\delta\left(\frac{1}{r_2}\right)$ &c.

Art. 62. Prop. *To apply the condition for aplanatism to the case of the double object-glasses of telescopes.*

The aperture of the object-glass of a telescope is so small in comparison of its focal length, that we have no need to go higher than the second approximation, in forming the equation

$$\Sigma\left(\delta\cdot\frac{1}{v}\right)=0$$

Referring to Art. 35, page 81, we have generally

$$\left(\delta\cdot\frac{1}{v}\right)=xr\cdot\frac{\mu-1}{\mu^2}A$$

$$=\frac{xr}{\mu f^3}\left\{(3\mu+2)m^2+(\mu+2)n^2+4(\mu+1)mn-\frac{\mu(3\mu+1)}{\mu-1}m-\right.$$
$$\left.-\frac{\mu(2\mu+1)}{\mu-1}n+\frac{\mu^3}{(\mu-1)^2}\right\}$$

If we put μ_1, m_1, n_1, p_1, f_1, and A_1 for the values of μ, m, n, p, f, and A for the first lens; and μ_2, m_2, n_2, p_2, f_2, and A_2 for the second lens, we have

$$\Sigma\left(\delta\cdot\frac{1}{v}\right)=0=\frac{\mu_1-1}{\mu_1^2}A_1+\frac{\mu_2-1}{\mu_2^2}A_2$$

since $y^2=2xr$ is nearly the same for the two lenses.

We have also the equations, see PART I. Art. 85,

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

$$0=\frac{p_1}{f_1}+\frac{p_2}{f_2}$$

and we have to determine the radii r_1, s_1, r_2, s_2, respectively, of the surfaces of the lenses; or, which is the same thing, we have to find f_1 and f_2, with n_1 and n_2, when m_1 is given, and therefore m_2 is easily found in terms of m_1 and f_1.

Now f_1 and f_2 are given by the two last equations, and we have only the one equation for *aplanatism* to determine n_1 and n_2; that is, one of these is arbitrary.

Amongst the different conditions which have been proposed, in order to furnish another equation, that of Sir John Herschel is undoubtedly *theoretically* the best; the condition proposed by the celebrated French mathematician Clairaut, and adopted by Professor Barlow, that the contiguous surfaces of the lenses should have the same radii, convex and concave respectively, and be cemented together in order to save the light which is lost by

reflexion at those surfaces, has not been adopted for the object-glasses of telescopes, although it is in general use for those of microscopes.

Sir John Herschel's condition is this: the telescope being used to view very distant objects, we take $u_1 = \infty \therefore m_1 = 0$ and having $u_2 = -f_1$, $m_2 = \frac{f_2}{u_2} = -\frac{f_2}{f_1}$ which is therefore known, and with these we obtain the equation

$$0 = \frac{\mu_1 - 1}{\mu_1^2} . n_1 + \frac{\mu_2 - 1}{\mu_2} . n_2$$

with n_1, n_2, and given quantities, but the telescope may be required to view objects which are nearer, although u is still large, say more than ten times F, and m_1 so small that we may neglect m_1^2 in forming the condition of aplanatism. We thus obtain another equation with n_1, n_2 and known quantities, which is merely the coefficient of m_1 in the general expression, equated to zero, because the term with m_1^2 is neglected as very small, and the term independent of m_1 was previously equated to zero, by taking first $m_1 = 0$. The resulting object-glass is free from aberration for the heavenly bodies and for terrestrial objects of considerable proximity.

Sir John Herschel has given a valuable table for facilitating the construction of object-glasses on these conditions; but the practical opticians find the construction difficult, because the aberrations of the separate lenses are very large, and therefore the slightest deviation from the theoretical curvatures produces a very sensible error in the combination.*

A similar table, calculated with one of the conditions as before $m_1 = 0$, and the other that one of the lenses should have the *minimum* aberration, would probably be the most serviceable to the working optician, and enable him to furnish more effective instruments than Sir John Herschel's rules.

* See a little Treatise, entitled "Practical Illustrations of the Achromatic Telescope. Being the substance of two Papers read before the Society of Arts." By Mr. Ross. London, 1810.

ART. 63. PROP. *To investigate the aplanatism of the object-glasses of microscopes.*

We have to determine, as in the last Article, r_1, s_1, r_2, s_2, or their equivalents f_1, f_2, n_1, n_2, when u_1 is small, and therefore m_1 is large. Our equations as before, are

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

$$0=\frac{p_1}{f_1}+\frac{p_2}{f_2}$$

$$0=\frac{\mu_1-1}{\mu_1^2}A_1+\frac{\mu_2-1}{\mu_2^2}A_2$$

An additional equation would arise from the method hitherto practically used on the proposal of Clairaut, that is

$$r_2=-s_1$$

and the cementing the contiguous surfaces with Canada balsam, has been attended with no inconvenience, on account of the unequal expansions of the lenses by heat, because they were always small.

These equations suffice to find the form of an achromatic lens to be used as the power of a microscope which shall also be aplanatic; when the length of the tube of the instrument and the *power* of the object-glass are given.

It was stated at page 151, PART I. that the powers of the best microscopes are now made with three achromatic lenses, which for a series of years were each plane on the side towards the object. A later improvement has been effected by Mr. Lister in making the one furthest from the object triple; that is, two convex lenses of plate-glass, with a flint-glass lens between them, all cemented together.

The object gained by using three achromatic lenses for the power, was stated to be the large quantity of light collected in the image, which being also very accurately achromatic and aplanatic, allowed it to be highly magnified and still remain bright and distinct.

If we examine the expressions for A_1 and A_2, Art. 62, we see that m_1 rises to the square in the equation

$$0 = \frac{\mu_1 - 1}{\mu_1^2} A_1 + \frac{\mu_2 - 1}{\mu_2^2} A_2$$

and for any values of r_1, s_1, r_2, s_2, there will be generally two values of m_1 (or none), which will satisfy the equation; and therefore (unless when the two roots are equal) there will be *two positions* of the luminous point, which will have aplanatic conjugate foci, if there be *one* such.

Again, at these foci, although the direct aberration may vanish, the oblique aberration will not necessarily do so, but will remain of some degree of magnitude, unless the compound lens has a particular form. The degree and direction of this *residual* oblique aberration depends on the excess of effect by the *convex* or *concave* lens, according to the particular form of each, and the direction of the pencil through it.

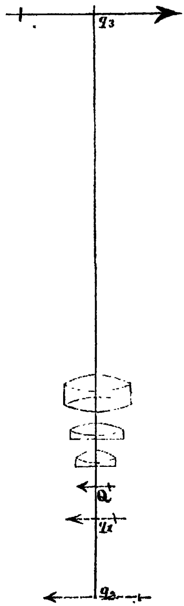
Mr. Lister discovered experimentally the two aplanatic foci of his plano-achromatic lenses, which are shewn to exist *generally*, from the analysis above, and also the effect on the oblique aberration in them, which he describes in the following terms:—"One other property of the double object-glass remains to be mentioned; which is, that when the longer aplanatic focus is used, the marginal rays of a pencil not coincident with the axis of the glass are distorted, so that a coma is thrown outwards, while the contrary effect of a coma directed towards the centre of the field is produced by the rays from the shorter focus. These peculiarities of the coma seem inseparable attendants on the two foci, and are as conspicuous in the achromatic meniscus, as in the plano-convex object-glass."*

We see that Mr. Lister thought the oblique aberration an inseparable attendant of the aplanatic foci, but now that our analysis is brought to bear on the subject, we see that a single lens may be free from the effect of obliquity, and therefore a double one

* See Phil. Trans. for 1830, page 197.

composed of two such, see Art. 46, and as there are generally two values of n_1 and n_2, which satisfy the equation of aplanatism for a given value of m_1, we must evidently in the *cemented* double achromatic lens take the value of n_1, which gives the outer surfaces the nearest in accordance with those of the single lens, which is free from the effect of obliquity.

The annexed figure will explain the positions of the successive foci in the triple achromatic power of a microscope.



The object Q is placed so near the lower lens, that the *virtual* image it forms, will be at some such position as q_1; the rays fall on the second lens as if they had come from an object at q_1, which is still so near the lens as to give a second *virtual* image as at q_2, but the rays falling upon the uppermost of the object-lenses, as if coming from an object so distant as q_2, give a *real* image as at q_3, which being again magnified by the eye-piece, we have the compound microscope completed.

The position of Q is so near the first lens, that the incident pencil forms a cone of large angle (about 90° in the figure), and the second lens must be large enough to transmit it also, and similarly with the third lens; so that if the object Q be bright we shall have a bright image at q_3, and if it be also very accurately achromatic and aplanatic, it will bear magnifying many times again by the eye-piece.

Mr. Lister having his lower lenses plano-achromatics, was under the necessity of using, approximately, the aplanatic foci which belong to such lenses, but he had still at command the focal lengths and distances, so that by changing these, the final correction could be made very complete as to achromatism, as well as direct and oblique aplanatism. He says: "The adjustment

for the microscope is then perfected, if necessary, by slightly varying the distance between the object-glasses, and after that is done, the length of the tube which carries the eye-pieces may be altered greatly without disturbing the correction; opposite errors which balance each other being produced by the change.

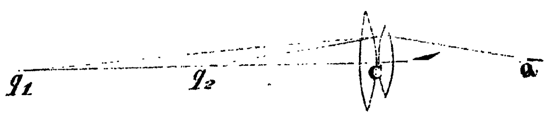
"In combining several glasses together, it is often convenient to transmit an under-corrected pencil from the front glass, and to counteract its error by over-correction in the middle one.

"Slight errors in colour may in the same manner be destroyed by opposite ones; and on the principles described, we not only acquire fine correction for the central ray, but by the opposite effects at the two foci on the transverse pencil, all coma can be destroyed, and the whole field rendered beautifully flat and distinct."

When a thin plate of glass is placed over the object, its aberration, which depends on the thickness, see Art. 24, will be in addition to those of the lenses, and hence a different separation of the lenses is required for aplanatism; this is accomplished by an appropriate mechanism according to Mr. Ross's improvement, which is now universally employed in the finest English microscopes, and this adjustment is an essential point to be attended to when viewing the most difficult test-objects.

Art. 64. Prop. *To put the equations for a double achromatic and aplanatic lens in a form for use in computation.*

In the first instance, until the forms of the lenses corresponding to some given position of the object at Q are determined, we take



the formulæ as applying to such a case as the annexed figure;

and have $$0=\frac{\mu_1-1}{\mu_1^2}A_1+\frac{\mu_2-1}{\mu_2^2}A_2 \quad \ldots\ldots\ldots \quad (1)$$

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$0=\frac{p_1}{f_1}+\frac{p_2}{f_2} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Let ϖ = the dispersive ratio $=\frac{p_1}{p_2}$ which must be given, then

from (3) $$\frac{1}{f_2}=-\frac{\varpi}{f_1}$$

substituting this in (2)

$$\frac{1}{F}=\frac{1-\varpi}{f_1}$$

$$\therefore \quad f_1=(1-\varpi)F$$

and $$f_2=-\frac{f_1}{\varpi}=-\frac{1-\varpi}{\varpi}F$$

which give f_1 and f_2 in terms of F the focal length of the compound lens and the dispersive ratio.

Again, $u_1=QC, \quad -u_2=q_1C=v_1$

$$\frac{1}{v_1}=\frac{1}{f_1}-\frac{1}{u_1}$$

$$=\frac{1-m_1}{f_1}$$

$$=-\frac{1}{u_2}$$

$$=-\frac{m_2}{f_2}$$

$$\therefore \quad m_2=-\frac{f_2}{f_1}(1-m_1)$$

$$=+\frac{1-m_1}{\varpi}$$

Substituting in the expression for A, Art. 62, the particular values, we have (1) in the form

$$0=\frac{1}{\mu_1 f_1^3}\Big\{(3\mu_1+2)m_1^2+(\mu_1+2)n_1^2+4(\mu_1+1)m_1n_1-$$

$$-\frac{\mu_1(3\mu_1+1)}{\mu_1-1}m_1-\frac{\mu_1(2\mu_1+1)}{\mu_1-1}n_1+\frac{\mu_1^3}{(\mu_1-1)^2}\Big\}$$

$$+\frac{1}{\mu_2 f_2^3}\Big\{(3\mu_2+2)\Big(\frac{1-m_1}{\varpi}\Big)^2+(\mu_2+2)n_2^2+4(\mu_2+1)n_2\Big(\frac{1-m_1}{\varpi}\Big)-$$

$$-\frac{\mu_2(3\mu_2+1)}{\mu_2-1}\Big(\frac{1-m_1}{\varpi}\Big)-\frac{\mu_2(2\mu_2+1)}{\mu_2-1}n_2+\frac{\mu_2^3}{(\mu_2-1)^2}\Big\}$$

which rises only to the second degree in m_1 n_1 n_2, and gives in general *two* values of any one of these quantities, when the other two are given.

The above equation may be simplified by multiplying up μ_1, f_1^3 and putting $\kappa=$ the refractive ratio $=\frac{\mu_1}{\mu}$

then $$\frac{\mu_1 f_1^3}{\mu_2 f_2^3}=\kappa\varpi$$

When the inner surfaces coincide for cementing, we have

$$s_1=-r_2$$

or $$\frac{f_1(\mu_1-1)}{1-n_1(\mu_1-1)}=\frac{f_2}{n_2}$$

$$\therefore\quad n_2=\frac{1-n_1(\mu_1-1)}{\varpi(\mu_1-1)}$$

Ex. 1. Let $-v=2u_1$ for the lower lens of the object-glass of the achromatic microscope, and $\varpi=\frac{1}{0.567}=1.764$ being the reciprocal of the dispersive ratio used by Sir John Herschel for his Ex. of a telescopic lens, since we place the concave flint lens nearest the object; and $\mu_1=1.589$, $\mu_2=1.519$ from his example,

then $$-\frac{1}{v}=-\frac{1}{2u_1}=\frac{1}{F}-\frac{1}{u_1}$$

$$\frac{1}{u_1}\left(1-\frac{1}{2}\right)=\frac{1}{F}$$

$$u_1=\frac{F}{2}=\frac{f_1}{2(1-\varpi)}$$

$$=\frac{f_1}{m_1}$$

$$\therefore \quad m_1=2(1-\varpi)$$

$$=-2\times .764=-1.528$$

and m_1 is here correctly negative, for u_1 is positive and f_1 negative,

and
$$m_2=\frac{1-m_1}{\varpi}$$

$$=\frac{2.528}{1.764}$$

$$=1.433$$

$$n_2=\frac{1-n_1(\mu_1-1)}{\varpi(\mu_1-1)}$$

$$=\frac{1-.589n_1}{.589\times 1.764}$$

$$=+.963-.567\,n_1$$

$$\kappa\varpi^3=-\frac{\mu_1}{\mu_2}\cdot\frac{f_1^3}{f_2^3}$$

$$=5.742$$

Substituting these values, we find the following quadratic equation for n_1,

$$0=2.907\,n_1^2-3.503\,n_1+.294$$

$$\therefore \quad n_1=.6025\pm\sqrt{.262}$$

$$=1.114 \text{ or } .090$$

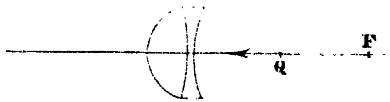
Taking the first value of n_1 we have

$$r_1=\frac{f_1}{n_1}=\frac{1-\varpi}{n_1}F$$

$$=-.686\,F$$

$$s_1 = \frac{(\mu_1 - 1) f_1}{1 - (\mu_1 - 1) n_1}$$
$$= -1.308\, F$$
$$r_2 = -s_1$$
$$= +1.308\, F$$
$$s_2 = \frac{(\mu_2 - 1) f_2}{1 - (\mu_2 - 1) n_2}$$
$$= .2714\, F$$

and the lens is as the figure drawn to the scale, F = one inch.



If we take the other value of n_1, we have

$$r_1 = -8.49\, F$$
$$s_1 = -.475\, F$$

and the first surface is much more nearly plane than in the former case; which approximates to the form of the meniscus in Ex. 5, Art. 46, with the oblique aberration the same as the direct, and which we should consequently adopt in practice.

Ex. 2. To find the form of the lens, when $v = 20u$ for the furthest of the three from the object, which has its conjugate focus real.

We have $$\frac{1}{v} = \frac{1}{20u} = \frac{1}{F} - \frac{1}{u}$$

whence
$$m_1 = -.727$$
$$m_2 = .979$$
$$n_2 = .963 - .567\, n_1$$
$$\varkappa \varpi^3 = 5.742$$

and with these we find that the condition of aplanatism cannot be satisfied, since the values of n_1 in the equation

$$0 = \frac{\mu_1 - 1}{\mu_1^2} A_1 + \frac{\mu_2 - 1}{\mu_2^2} A_2$$

are imaginary, but if we equate to z a small quantity in place of zero, we find,

$$z = 2.907n_1^2 + 3.097n_1 + .851$$

$$\therefore \quad n_1 = -.533 \pm \sqrt{\frac{z}{2.907} - .009}$$

and supposing the quantity under the radical sign to vanish,

we have
$$n_1 = -.533$$

$$r_1 = \frac{f_1}{n_1} = \frac{1-\varpi}{n_1} F$$

$$= + \frac{.764}{.533} F$$

$$= 1.433\, F$$

$$s_1 = \frac{(\mu_1 - 1) f_1}{1 - (\mu_1 - 1) n_1}$$

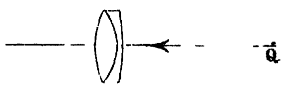
$$= -.342\, F$$

$$r_2 = -s_1 = .342\, F$$

$$s_2 = \frac{(\mu_2 - 1) f_2}{1 - (\mu_2 - 1) n_2}$$

$$= .654\, F$$

and the lens to a scale $F =$ one inch, is as in the figure.

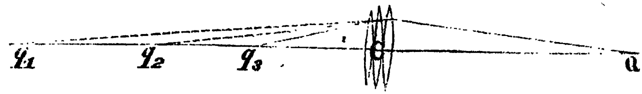


This Example shews us that Clairaut's condition, of the inner surfaces coinciding, is not always compatible with aplanatism in the double lens.

Art. 65. Prop. *To investigate the equations for finding the form of a* triple *achromatic and aplanatic lens.*

We suppose the two outer lenses to be of the same kind of glass, crown or plate, and the middle one to be of flint-glass.

We suppose the three lenses and the positions of the successive foci to be as in the figure, in the first instance, and until the



circumstances for any particular case are determined; and our equations for achromatism and aplanatism are as follows,

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3} \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$0=\frac{p_1}{f_1}+\frac{p_2}{f_2}+\frac{p_1}{f_3} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$0=\frac{\mu_1-1}{\mu_1^2}A_1+\frac{\mu_2-1}{\mu_2^2}A_2+\frac{\mu_1-1}{\mu_1^2}A_3 \ldots\ldots \quad (3)$$

We have to determine the six radii r_1, s_1, r_2, s_2, r_3, s_3, or their equivalents f_1, f_2, f_3, n_1, n_2, n_3; so that we need three more equations. These three equations may be formed on any conditions which establish desirable properties in the compound lens, provided they do not lead to results incompatible with the above three equations.

If the lenses are to be cemented for a microscope object-glass, we have

$$r_2=-s_1$$

$$r_3=-s_2$$

or

$$\frac{f_2}{n_2}=\frac{f_1(\mu_1-1)}{1-n_1(\mu_1-1)} \quad \ldots\ldots\ldots \quad (4)$$

$$\frac{f_3}{n_3}=-\frac{f_2(\mu_2-1)}{1-n_2(\mu_2-1)} \quad \ldots\ldots\ldots \quad (5)$$

To form our remaining equation, we may suppose the two outer surfaces to have their radii in the same proportion as those of a single lens in which the effect of obliquity vanishes, as found by the rule of Art. 46, from the expression

$$n=\frac{\mu^2-m(2\mu^2-\mu-1)}{\mu^2-1}$$

where m is to be found from the given value of F and the given

distances $QC=u_1$, $q_3C=v_3$; μ being the average refractive index for the lens, such that,

$$(\mu-1)\left(\frac{1}{r_1}+\frac{1}{s_3}\right)=\frac{1}{F}$$

then
$$m=\frac{F}{u_1}$$

$$r_1=\frac{F}{n}$$

$$=\frac{F(\mu^2-1)}{\mu^2-\frac{F}{u_1}(2\mu^2-\mu-1)}$$

$$s_3=\frac{(\mu-1)F}{1-(\mu-1)n}$$

$$=\frac{(\mu^2-1)F}{\mu+1-\mu^2+\frac{F}{u_1}(2\mu^2-\mu-1)}$$

Our sixth equation, therefore, takes the form $s_3=\gamma r_1$.

For substitution in the equation of aplanatism, we have from the two first equations of this Article, and remembering that $n_1r_1=f_1$,

$$\frac{1}{f_1}=\frac{1}{n_1r_1}$$

$$\frac{1}{f_2}=-\frac{\varpi}{(1-\varpi)F}$$

$$\frac{1}{f_3}=\frac{1}{(1-\varpi)F}-\frac{1}{n_1r_1}$$

From the fourth and fifth equations, we have

$$n_2=-\frac{F(1-\varpi)}{\varpi r_1}\left(1-\frac{1}{n_1(\mu_1-1)}\right)$$

$$n_3=\frac{n_1\left(\frac{\varpi r_1}{\mu_2-1}+(1-\varpi)F\right)-\frac{(1-\varpi)F}{\mu_1-1}}{n_1r_1-(1-\varpi)F}$$

Again, from the positions of the successive foci, we have

$$m_2=(1-m_1)\frac{(1-\varpi)}{n_1r_1\varpi}\cdot F$$

$$m_3=\frac{n_1r_1\varpi-(1-m_1)(1-\varpi)F}{n_1r_1-(1-\varpi)F}$$

These quantities being substituted in the equation of aplanatism

$$0=\frac{\mu_1-1}{\mu_1^2}A_1+\frac{\mu_2-1}{\mu_2^2}A_2+\frac{\mu_1-1}{\mu_1^2}A_3$$

we have a numerical equation for determining the remaining unknown quantity n_1.

Art. 66. Prop. *To investigate the equations for finding the form of a double achromatic lens which shall be* aplanatic *to the* third approximation.

The equation which gives the direction of a ray refracted by

a lens to a *third* approximation is in Art. 10 found of the form

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}+\frac{xr(\mu-1)}{\mu^3}A+\frac{x^2r^2}{f^5}D$$

Using the same notation as before, we have for the double lens, as follows

$$\frac{1}{v_1}=\frac{1}{f_1}-\frac{1}{u_1}+\frac{xr(\mu_1-1)}{\mu_1^3}A_1+\frac{x^2r^2}{f_1^5}D_1$$

$$\frac{1}{v_2}=\frac{1}{f_2}+\frac{1}{v_1}+\frac{xr(\mu_2-1)}{\mu_2^3}A_2+\frac{x^2r^2}{f_2^5}D_2$$

$$=\frac{1}{f_2}+\frac{1}{f_1}-\frac{1}{u_1}+xr\left(\frac{\mu_1-1}{\mu_1^3}A_1+\frac{\mu_2-1}{\mu_2^3}A_2\right)+x^2r^2\left(\frac{D_1}{f_1^5}+\frac{D_2}{f_2^5}\right)$$

and if v_2 is the same for every ray, whatever may be the value of xr, we must have

$$0=\frac{\mu_1-1}{\mu_1^2}A_1+\frac{\mu_2-1}{\mu_2^2}A_2 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$0=\frac{D_1}{f_1^5}+\frac{D_2}{f_2^5} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Now D_2 must contain the coefficients of every term which, for the second lens, rises to x^2r^2, and therefore $\frac{1}{v_1}$ must be used to the second approximation in forming the value of A, which will thus yield A_2 and also terms which have x^2r^2,

thus
$$\frac{1}{v_1}=\frac{1-m_1}{f_1}+xr\frac{(\mu_1-1)}{\mu_1^2}A_1$$
$$=-\frac{1}{u_2}$$
$$=-\frac{m_2}{f_2}$$
$$\therefore \quad m_2=-\frac{f_2}{f_1}(1-m_1)-\frac{xr(\mu_1-1)f_2}{\mu_1^2}A_1$$
$$=\frac{1-m_1}{\varpi}+xr\frac{(\mu_1-1)(1-\varpi)F}{\mu_1^2\varpi}A_1$$

If the equation (1) be solved after m_1, μ_1, μ_2, ϖ, and F are expressed in it numerically, we shall have a relation between n_1 and n_2, and then the equation (2) suffices to determine them, and so also the form of the lens.

CHAPTER VI.

ON THE FORMS AND PROPERTIES OF EYE-PIECES.

THE use of an eye-piece of a telescope or microscope is to produce, in a position fit for vision, a magnified image, of the real image given by the object-glasses; and distinctness extended over a large field of view is the desirable property to be sought for. Now, the properties of lenses which produce indistinctness are *the chromatic dispersion, the direct and oblique aberrations, the confusion* in excentrical pencils, and *the curvature* of the image which allows only certain portions to remain within the limits of distance required for distinct vision by the eye.

The use of a single eye-lens for very critical astronomical observations, except in all those cases where a large field is absolutely necessary, has the following recommendation from the pen of Sir Wm. Herschel, whose experience and success leave his opinion beyond gainsaying.*

"Let us resign the double eye-glass to those who view objects merely for entertainment, and must have an exorbitant field of view. To a philosopher this is an unpardonable indulgence. I have tried both the single and double eye-glass of equal powers, and always found that the single eye-glass had much the superiority in point of light and distinctness. With the double eye-glass I could not see the *belts on Saturn*, which I very plainly saw with the single one. I would, however, except all those cases where a large field is absolutely necessary, and where power joined to distinctness is not the sole object of our view."

The equi-convex is the lens which has generally been used as a single eye-glass. The prismatic effect of a thin lens on a ray

* See Phil. Trans. vol. LXXII, pp. 94, 95.

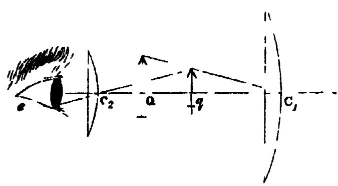
passing through it at a given distance from the axis, we saw in Art. 59, depended on the focal length, and remained the same when the focal length was the same, however the radii of the surfaces might be changed, so that in such a lens the chromatic dispersion cannot be removed or reduced, with the same magnifying power.

When the lens is thick, a small effect like that in the achromatic eye-piece takes place.

The oblique and direct aberrations can, however, be greatly modified and reduced by changing the form of the lens, and from Ex. 9, page 85, Ex. 4, page 119, and Ex. 2, page 142, we should expect the plano-convex lens, with the convex surface turned towards the eye, to possess advantages sufficient to ensure its use. To shew the boundaries of the field of view distinctly, however, the lens should be of the form found below for the field-glass of Huygens's eye-piece.

Art. 67. Prop. *To investigate the origin of the advantages obtained in Huygens's eye-piece.*

The figure represents Huygens's eye-piece with both lenses convexo-plane, and the focal length of the field-glass three times that of the eye-glass, with the distance between them equal to twice the focal length of the eye-glass.



The condition of achromatism as investigated in Art. 60, is satisfied exactly.

From Ex. 8, page 85, we see that the direct pencil from the object-glass falls in a very favourable manner, though not the most so, on the field-glass for small aberration.

If we make use of the result of Art. 50, page 128, as found in

the Examples, in order that the excentrical pencils may have their angles of incidence and emergence equal, and therefore the confusion very small, we have $r=\frac{2}{3}f$, $s=2f$. The convexo-plane is an approximation towards this form, and hence the confusion of the field-glass is small in the parts of the image out of the axis.

If we examine the conditions, in order that the lens shall give an image of continued curvature in accordance with Art. 53 and Art. 46, we find that the given positions of the conjugate foci render our expressions unintelligible, but Ex. 3, page 143, shews us that the proper form for the field-glass is a meniscus with the concave surface towards the eye, and the image of a straight line is *concave* to the eye-glass.

From these considerations, we perceive that the convexo-plane field-glass is an excellent compromise to approximate to several desirable but incompatible qualities, at the same time; although no one of these is fully acquired.

Again, the eye-glass is not of the best form for shewing the central parts of the field of view the most distinctly, which, from Ex. 2, page 83, and Ex. 4, page 119, we see should have the most curved side towards the eye; but applying the condition that the excentrical pencils should have equal angles of incidence and emergence, we find, as in Ex. 2, page 130, that a deep meniscus with radii $r=\frac{2}{5}f$, $s=-2f$ is the proper form, with the convex surface towards the incident light.

We see that the convexo-plane lens is intermediate and a compromise between the two forms, though much nearer to the latter, and gives moderately distinct vision over a large field, which the concave image furnished by the field-glass keeps within the limits of the distance required by the eye.

Art. 68. Prop. *To investigate the origin of the advantages obtained in Ramsden's eye-piece.*

The lenses of Ramsden's eye-piece being, as in the figure, two lenses of equal focal lengths set at the distance of about two-thirds the focal length of either, the field-glass is plano-convex, and the eye-glass convexo-plane.

The lenses are set too near together to satisfy completely the condition of achromatism; and if further separated, the imperfections and unavoidable dust on the field-glass would be seen with the image, magnified by the eye-glass; but we shall see also that it is requisite, the real image formed by the object-glass should be at some distance from the field-glass, in order that its virtual image may be sufficiently concave to the eye-glass to produce an extended field of view, within the range of distinct vision.

By the result of Ex. 9, page 85, we see that the field-glass is nearly in the position for its minimum aberration; and by the results of Ex. 5, page 120, we see that the plano-convex lens approximates to the meniscus, which is the correct form, in order that the oblique aberration may be the same as the direct.

A deep meniscus we saw also in Ex. 5 *bis*, page 145, gives an image of a straight line which is concave to the lens in its central parts; but as the pencils forming the other parts of the image fall very excentrically on the field-glass, we must examine the form and distinctness of those parts as given by the primary and secondary foci.

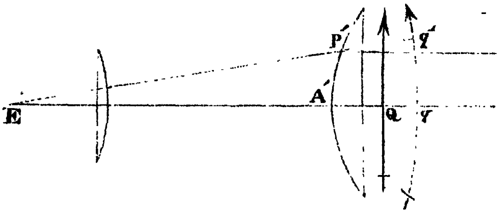
To apply the expressions of Art. 51, page 133, to the case of the field-glass of Ramsden's eye-piece, we have $r = \infty$, $c =$ the focal length of the object-glass of the telescope nearly, and consequently very large compared with f that of the field-glass; $b = f$, nearly, and we may suppose t the thickness small compared with f.

Since the image formed by the object-glass is near the field-glass, we have u small compared with s and f, and need in the small term retain only the quantities which have u^2 in the denominator, rejecting also those which have t in the numerator even

when they have u^2 in the denominator. We have then the expressions for $\frac{1}{v_1}$ and $\frac{1}{v_2}$ identical, or the confusion is small in these circumstances,

thus $$\frac{1}{v_1}=\frac{1}{v_2}=\frac{\mu-1}{s}-\frac{1}{u}+\frac{t}{\mu\,.\,u^2}-\frac{\theta^2 b^2}{2\mu u^2 s}$$
$$=\frac{1}{v}-\frac{\theta^2 b^2}{2\mu u^2 s} \quad \text{say}$$

where v_1, v_2 and v are negative, and the image virtual, as at q, q' in the figure; they are also to be measured from the point P'.



Taking E for the pole, let $Eq'=\rho=EP'+P'q'$

and $$P'q'=v_2$$
$$=v-\frac{\theta^2 v^2 f^2}{2\mu u^2 s}$$

$$EP'=f+\frac{\theta^2 f(f+s)}{2s}$$ from the approximate equation to the circular arc $A'P'$.

$$\therefore \quad \rho=f+v+\frac{\theta^2 f}{2s}\left\{f+s-\frac{v^2 f}{u^2 \mu}\right\}$$

and putting for s its value $(\mu-1)f$, also neglecting t in the value of v, we have $v=\frac{fu}{f-u}$

We have $$\rho=\frac{f^2}{f-u}+\frac{\theta^2 f}{2\mu(\mu-1)}\left\{\mu^2-\frac{f^2}{(f-u)^2}\right\}$$

We see that the coefficient of θ^2 diminishes as u from being very

small, is increased; and by comparing with the approximate equation of a straight line, namely,

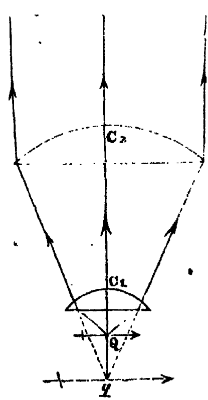
$$\rho' = a + \frac{\theta^2 a}{2}$$

we find that the virtual image becomes concave to the lens when u is increased, for $\mu = \frac{3}{2}$ beyond the value $.146 f$, and hence, in order to see distinctly a large extent of the micrometer wires and image formed by the object-glass, they must not be too close to the field-glass of the eye-piece, which is according to the rule adopted by Ramsden.

The result of the discussions of Huygens's and Ramsden's eye-pieces shews us, that if any improved eye-piece should be found, it will be by taking those forms of lenses which secure the most desirable properties, whilst the less desirable ones are neglected.

ART. 69. PROP. *To investigate the origin of the advantages in* Wollaston's doublet.

As explained at page 152, PART I., Wollaston's doublet consists of two lenses, of the same forms, focal lengths, and distance, as constitute a Huygens's eye-piece, being used for a microscope with the lens of the longer focus turned towards the eye.



In the annexed figure let f_1 be the focal length of the lens c_1, f_2 that of c_2, and the distance $c_1, c_2 = \frac{1}{2}(f_1 + f_2)$; also $f_2 = 3 f_1$.

Let Q be the place of the object, and q that of its virtual image given by the lens c_1. The primary image q is viewed through the eye-lens c_2 as if it were an object; and the virtual image seen by the eye must be taken as at the least distance of distinct vision.

We may, however, consider the rays emerging from c_2 as parallel, since the focal lengths of the lenses are very small compared with

the least distance of distinct vision; the combination being equivalent to a single lens of $\frac{1}{20}$th, $\frac{1}{30}$th, $\frac{1}{40}$th, or $\frac{1}{50}$th inch focus, frequently.

Then we have $$c_2q=f_2=3f_1, \quad \text{nearly}$$
$$c_1q=c_2q-c_1c_2=3f_1-\frac{1}{2}(f_1+f_2)$$
$$=f_1 \quad \text{nearly}$$
again, for the conjugate foci of the lens c_1 we have
$$-\frac{1}{c_1q}=\frac{1}{f_1}-\frac{1}{c_1Q}$$
$$=-\frac{1}{f_1}$$
$$\therefore \quad c_1Q=\frac{1}{2}f_1$$

and the object-lens magnifies the image twice, which is equivalent to doubling the power of the eye-lens if taken by itself. This coincides with the result obtained by using the rule found in Art. 81, PART I.

Both the lenses are used in favourable circumstances for small aberrations, as shewn in Ex. 9, page 85, and they also approximate to the proper forms in respect of the oblique aberrations, as shewn in Ex. 4 and Ex. 5, pages 119 and 120. It appears, however, that a meniscus is the more appropriate form for the lower lens, both from Ex. 5, page 120, and from page 90; for the eye-lens, however, a meniscus would have had less direct aberration but more oblique aberration, as seen from Ex. 4, page 119. The curvature of the virtual image q would also be more favourable with a meniscus in place of the plano-convex object-glass, as shewn by Ex. 5, page 145.

The distance required between the lenses, in order to procure achromatism, from page 160, should be
$$c_1\, c_2=\frac{f_1+f_2}{2-\dfrac{f_1}{u}}$$

and since $u = \frac{f_1}{2}$ as found above, $\therefore c_1 c_2 = \infty$ for achromatism, which consequently is not satisfied.

The effect of the diaphragm introduced by the practical opticians, is to limit the direct pencil and also to cut off the extreme outside rays of the oblique pencils, and thus contribute to the distinctness of the parts of the image formed by them.

CHAPTER VII.

ON CAUSTICS.

IN Article 18, PART I., it was shewn how the caustic curve in light reflected by a spherical mirror arose from the continual intersection of consecutive rays, and was connected with the aberration. Whenever aberration exists, it is clear that a caustic will be formed, both in refracted and reflected light. The term caustic curve has been most frequently reserved for the locus of the ultimate intersections of the rays in the plane of incidence, and termed a catacaustic for reflexion, and a diacaustic for refraction.

When the reflecting or refracting surface is one of revolution and the pencil incident directly, the revolution of the caustic round the same axis forms a caustic surface. This surface being the locus of the primary foci (see page 25), we may call it the primary caustic surface. We have seen, however, that consecutive rays also intersect at the secondary focus, which is always in the line drawn through the luminous origin and the center for a spherical surface, so that the assemblage of secondary foci forms a straight line, which is also rightly denominated a caustic. In other forms of reflecting or refracting surfaces, the assemblage of secondary foci will generally form a secondary caustic surface. In aplanatic reflexion or refraction these surfaces are reduced to one point, which is the aplanatic focus, and at spherical surfaces we have seen that the secondary caustic surface is reduced to a line.

That generally there will be two caustic surfaces, as shewn by Malus in vol. II. of the "*Mémoires des Savans Etrangers,*" we may conclude from the following considerations. When rays of light diverge in all directions around a luminous point, they will be normals to every spherical surface which has that point for center; or their orthogonal trajectory is everywhere a spherical surface.

When there is aplanatic reflexion or refraction, the orthogonal trajectory of the reflected or refracted rays will be, in every position, a spherical surface with the focus for center. In other cases the orthogonal trajectory of the reflected or refracted rays will be some other form of surface, and the primary and secondary foci of the reflected or refracted pencil will be the centers of greatest and least curvature of the trajectory; being the intersections of the consecutive normals to that surface.

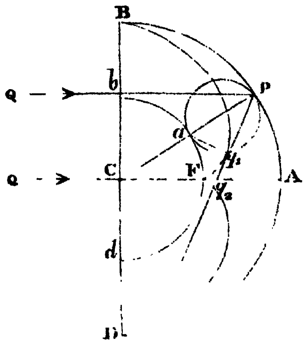
The discussion of the properties of systems of rays, as undertaken by Malus in the before-mentioned paper and by Sir William Hamilton in the "Transactions of the Royal Society of Dublin, 1828," is an interesting subject in the higher geometry, and the properties of caustics and of the orthogonal trajectories are also of importance in physical optics, from the interference of light which takes place in the trajectories when they form an edge of regression. To this problem it is now admitted the theory of the rainbow must be referred; but in the immediate objects of geometrical optics, such as the theory of optical instruments, the properties of caustics and their forms for different reflecting and refracting surfaces are of small moment, so that only a few of the more ordinary cases will be here discussed.

Art. 70. Prop. *To shew that when parallel rays are incident on a concave spherical mirror, the primary caustic is an epicycloid generated by a point in the circumference of a circle, whose radius is one-fourth that of the mirror, whilst it rolls upon a circle concentric with the mirror and of half its radius.*

Let BAD be the spherical mirror whose center is C; let F be the principal focus and $baFd$ a semicircle with radius $AF=\dfrac{AC}{2}$; let BFD be the epicycloid generated by a point in the circumference of a circle whose diameter is AF rolling on bFd; and let QP be any one of the rays incident parallel to CA and reflected in Pq_1q_2; then, if q_1 is the primary focus, it is a point in the epicycloid.

Draw CaP and the rolling circle on aP as diameter, when it

touches the mirror at P. The angle of incidence (i) is $\angle QPC$, and equals the angle of reflexion $\angle CPq_1$; join aq_1.



Then from Art. 11, $\frac{1}{v_1}=\frac{2\ \text{sec.}\ i}{r}-\frac{1}{u}$

and here $\frac{1}{u}=0 \quad \therefore\ v_1=\frac{r}{2}\cos.\ i$

$$=Pa\cos.\ i$$

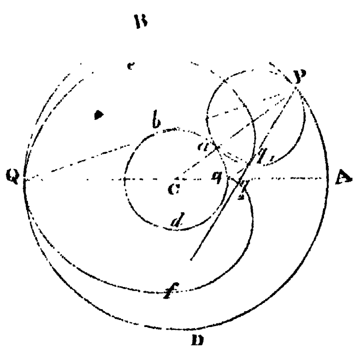
$$=Pq_1$$

or q_1 the primary focus is a point in the circumference of the circle aq_1P.

Again $\angle ACP=\angle QPC=\angle CPq_1$, and the angle subtended by the arc aq_1 at the center of the rolling circle equals twice the angle PCF, and the radii are in the ratio $1:2$, therefore the arc $Fa=$ arc aq_1, and q_1 is a point in the epicycloid Bq_1FD.

Art. 71. Prop. *To shew that when rays diverge from a point in the circumference of a spherical mirror, the primary caustic, formed by the reflected rays, is an epicycloid generated by a point in the circumference of a circle, of one-third the radius of the mirror whilst it rolls on a fixed circle concentric with the mirror, also of one-third its radius.*

Let $QBAD$ be the section of the mirror by the plane of incidence; Q the luminous point, and QP any incident ray reflected in Pq_1q_2.



Draw the radius CaP, and let $baqd$ be the fixed circle whose radius Ca or Cq equals one-third of CA. Also on aP as diameter, draw the rolling circle as in the figure, and let $Qeqf$ be the epicy-

cloid described by the point in its circumference. Let q_1 and q_2 be the primary and secondary foci. Then, angle of incidence $QPC = i = q_1PC$ the angle of reflexion. Join aq_1, by Art. 11,

$$\frac{1}{v_1} = \frac{2 \sec. i}{AC} - \frac{1}{QP}$$

$$= \frac{2}{AC \cos. i} - \frac{1}{2AC \cos. i}$$

$$= \frac{3}{2AC \cos. i}$$

$$Pq_1 = v_1 = \frac{2}{3} AC \cos. i$$

$$= aP \cos. i$$

and q_1 is in the circumference of the rolling circle.

Again, $\angle PCA = 2 \angle PQC = 2i =$ angle which the arc aq_1 subtends at the center of the rolling circle; and the radii of the fixed and rolling circles are equal, therefore arc $aq =$ arc aq_1, and the point q_1 is in the epicycloid.

In other positions of the luminous point, the caustic is not a known curve, but can always be drawn by calculating the positions of a succession of points. It can be shewn, as in the next Article, that in all cases where the aberration varies as the square of the semi-aperture either for reflexion or refraction, that the primary caustic has a cusp at the geometrical focus, and is ultimately a semi-cubical parabola. The method is that of Mr. Coddington.*

ART. 72. PROP. *To shew that when the aberration of a reflected or refracted pencil varies as the semi-aperture squared, the caustic is* ultimately *a semi-cubical parabola.*

Let the origin O be the geometrical focus; pq_1Op' the caustic, having a cusp at O; q_1 the intersection of two consecutive rays, one of which Pq_1q meets the axis at q.

* "Optics," p. 231.

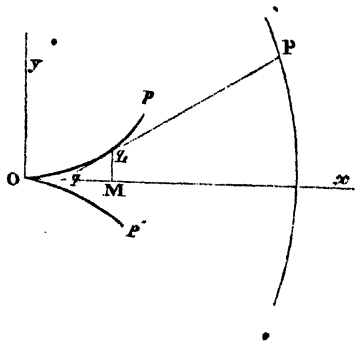
Let $OM=x$, $q_1M=y$ be the co-ordinates of q_1,

we have $\tan. q_1qM=\frac{dy}{dx}$

and Oq the aberration, varies as $\left(\frac{1}{2}\text{ aperture}\right)^2$

$$=a\left(\frac{dy}{dx}\right)^2 \text{ ultimately}$$

where a is a constant depending on the particular case.



Then
$$\begin{aligned} Oq &= OM - Mq \\ &= x - y\cot. q_1qM \\ &= x - \frac{y}{\frac{dy}{dx}} \\ &= a\left(\frac{dy}{dx}\right)^2 \quad \text{as above.} \end{aligned}$$

Or, we have to integrate the differential equation

$$x\frac{dy}{dx}-y=a\frac{dy^3}{dx^3}$$

differentiating, we have

$$\frac{dy}{dx}+x\frac{d^2y}{dx^2}-\frac{dy}{dx}=3a\frac{dy^2}{dx^2}\cdot\frac{d^2y}{dx^2}$$

or
$$\frac{d^2y}{dx^2}\left(3a\frac{dy^2}{dx^2}-x\right)=0$$

$$\therefore\quad \frac{d^2y}{dx^2}=0\,,\qquad \text{or}\qquad 3a\frac{dy^2}{dx^2}=x$$

The first being integrated, gives the equation of one of the rays; and the second gives, as a singular solution of the differential equation, the curve which is the locus of the ultimate intersections of the rays, or the caustic in its ultimate form near O.

for
$$\frac{dy}{dx}=\frac{x^{\frac{1}{2}}}{\sqrt{3a}}$$

$$\therefore \quad y=\frac{2}{3\sqrt{3a}}x^{\frac{3}{2}} \quad \text{by integration}$$

or
$$y^2=\frac{4}{27a}x^3 \quad \text{the equation of a semi-cubical para-}$$

bola.

The primary caustic, in many cases, can be readily found as the locus of the ultimate intersections of consecutive rays, by the ordinary method of singular solutions.

If $V=0$, be the equation of one of the refracted or reflected rays, which contains the co-ordinates x, y to the first power and constants, together with an indeterminate parameter m, and we eliminate m between the equations

$$V=0$$
$$\frac{dV}{dm}=0$$

the resulting equation is that of the caustic required. It is, however, generally easier to treat each case according to the nature of the data, without reducing to the form $V=0$ with only one indeterminate parameter.

Thus, let x, y be the co-ordinates of a point on the reflecting or refracting curve, whose equation is $f(x, y)=0$, the equation of the ray will be of the form

$$y'-y=A(x'-x)$$

where A is a function of x, y; and at the point of ultimate intersection x', y' remain constant whilst x, y, and A vary, therefore differentiating, we have

$$-\frac{dy}{dx}=\frac{dA}{dx}(x'-x)-A$$

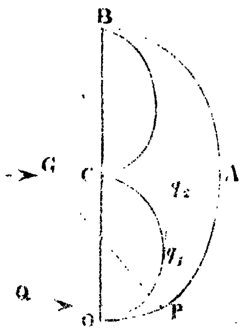
$$\therefore \quad x' = x + \frac{A \mp \frac{dy}{dx}}{\frac{dA}{dx}}$$

also $\qquad A = \frac{dy'}{dx'}$ since each ray is tangent to the caustic; and these equations give the solution from the integral

$$y' = \int A dx'$$

or otherwise, by eliminating x, and y from the equations.

ART. 73. PROP. *To find the caustic when rays parallel to the axis are reflected by a concave cycloidal mirror.*



Let $OPAB$ be the cycloid, Ox, Oy the co-ordinate axes, and the equation of the cycloid

$$y = a \text{ vers}^{-1} \cdot \frac{x}{a} - \sqrt{2ax - x^2}$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{x}{2a - x}}$$

Let the ray QP be incident at P, and reflected in the line Pq_1q_2, PG being the normal at P.

Now if i = angle of incidence $= \angle QPG = \frac{\pi}{2} - \tan^{-1} \cdot \frac{dy}{dx}$

and $\qquad y' - y = A(x' - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

be the equation of the reflected ray, we have

$$A = -\tan. 2i$$

$$= \frac{2\frac{dy}{dx}}{1 - \frac{dy^2}{dx^2}}$$

$$= \frac{\sqrt{2ax - x^2}}{a - x}$$

and $$\frac{dA}{dx}=\frac{a^2}{(a-x)^2\sqrt{2ax-x^2}}$$

differentiating the equation (1) considering x', y' the co-ordinates of the primary focus q_1 and therefore constant whilst x and y vary, we have, as on the last page,

$$x'=x+\frac{A-\frac{dy}{dx}}{\frac{dA}{dx}}$$

$$=\frac{2ax-x^2}{a}$$

$$\therefore\quad x=a\pm\sqrt{a^2-ax'}$$

and substituting, we have

$$A=\frac{dy'}{dx'}=\pm\sqrt{\frac{x'}{a-x'}}$$

taking the upper sign and integrating

$$y'=\int\sqrt{\frac{x'}{a-x'}}\cdot dx'$$

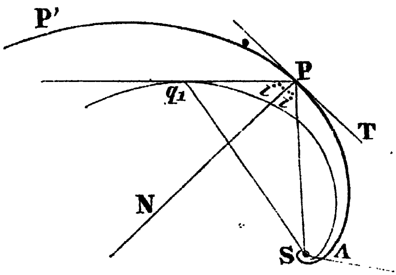
$$=\frac{a}{2}\text{vers}^{-1}\cdot\frac{2x'}{a}-\sqrt{ax'-x'^2}$$

the equation of a cycloid formed by a point in a rolling circle of half the radius of that describing the original cycloid.

Art. 74. Prop. *To find the form of the caustic when formed by the rays reflected at a logarithmic spiral, the luminous point being in the pole.*

Let S be the pole of the logarithmic spiral APP' whose equation is $r=a^\theta$, where SP, any incident ray of light, $=r$ and $ASP=\theta$.

Then if PT be the tangent at P, the angle SPT is constant by the property of the curve, and if PN be the normal, the angle of



incidence $(i) = SPN = \frac{\pi}{2} - SPT =$ constant,

or
$$\cot. i = \tan. SPT$$
$$= r\frac{d\theta}{dr}$$
$$= \frac{1}{\log_e a}$$

let q_1 be the primary focus, $Sq_1 = \rho$, and $\angle q_1SA = \phi$, we have in the triangle SPq_1

$$\frac{r}{\sin.(\pi - 2i - \phi - \theta)} = \frac{\rho}{\sin.(2i)}$$
$$\therefore \quad \rho = r\frac{\sin. 2i}{\sin.(2i + \phi - \theta)}$$

and for the locus of ultimate intersections ρ and ϕ remain constant whilst r and θ vary; therefore, differentiating,

$$0 = \frac{dr}{d\theta} \cdot \frac{\sin. 2i}{\sin.(2i + \phi - \theta)} + \frac{r \sin. 2i \cos.(2i + \phi - \theta)}{\sin.^2(2i + \phi - \theta)}$$

or $0 = \log_e (a) \,.\, a^\theta \sin. 2i \,.\, \sin.(2i + \phi - \theta) + a^\theta \,.\, \sin. 2i \cos.(2i + \phi - \theta)$

$$\therefore \quad \tan.(2i + \phi - \theta) = -\frac{1}{\log_e (a)}$$
$$= -\cot. i$$
$$= \tan.\left(i + \frac{\pi}{2}\right)$$

$$\therefore \quad \theta = i + \phi - \frac{\pi}{2}$$

$$= \phi - \alpha \qquad \text{if } \alpha = \frac{\pi}{2} - i = \text{constant}$$

and
$$\rho = a^{\theta} \cdot \frac{\sin. 2i}{\sin. (2i + \phi - \theta)}$$

$$= a^{\theta} \cdot \frac{\sin. 2i}{\sin. \left(\frac{\pi}{2} + i\right)}$$

$$= 2 . \sin. i . a^{(\phi - \alpha)}$$

which is the equation of another logarithmic spiral.

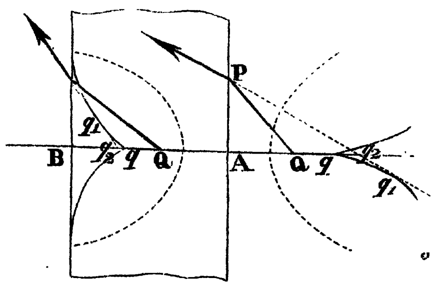
Let $\qquad 2 \sin. i = a^{\beta}$ then,

then $\qquad \rho = a^{(\phi + \beta - \alpha)}$

which shews that the caustic differs from the reflecting curve only in having a different origin of the angular ordinate.

Art. 75. Prop. *To find the form of the primary caustic when a diverging pencil of rays is refracted at a plane surface of a medium.*

First, let Q the luminous point be outside the medium, and $QA = u$.



Let q be the first approximate focus and the cusp of the caustic, the line $AQqq_2$ being perpendicular to the surface and the axis of x. Let A be the origin of co-ordinates, QP any ray incident at P, and $AP = y$; Pq_2q_1 being the direction of the refracted ray.

Then by Art. 28, Part I.,

$$q_2P=\mu \,.\, QP$$
$$=\mu\sqrt{u^2+y^2}$$

If the equation of the refracted ray be of the form

$$y'-y=A(x'-x)$$

we have, $$y=AP\,,\quad x=0$$

$$A=\tan.\,Pq_2x$$
$$=-\frac{AP}{q_2A}$$
$$=-\frac{y}{\sqrt{\mu^2u^2+y^2(\mu^2-1)}}$$

and differentiating the equation $y'-y=Ax'$ considering x', y' the co-ordinates of the point where the ray touches the caustic

then $$1=-\frac{dA}{dy}x'$$

or $$1=\frac{\mu^2u^2\,.\,x'}{\{\mu^2u^2+y^2(\mu^2-1)\}^{\frac{3}{2}}}$$

or $$x'=\frac{\{\mu^2u^2+y^2(\mu^2-1)\}^{\frac{3}{2}}}{\mu^2u^2} \qquad (1)$$

substituting the values of A and x' in the equation

$$y'=y+Ax'$$

we have $$y'=-\frac{y^3(\mu^2-1)}{\mu^2u^2} \qquad (2)$$

and eliminating y between (1) and (2) we have

$$\left(\frac{x'}{\mu u}\right)^{\frac{2}{3}}-\left(\frac{(\mu^2-1)^{\frac{1}{2}}\,y'}{\mu u}\right)^{\frac{2}{3}}=1$$

which is the equation of the caustic and is the evolute of an hyperbola.

Again, if we put $\frac{1}{\mu}$ for μ we have secondly, the result for the

luminous point Q within the dense medium, and the equation of the caustic becomes

$$\left(\frac{\mu x'}{u}\right)^{\frac{2}{3}} + \left(\frac{(\mu^2-1)^{\frac{1}{2}} \cdot y'}{u}\right)^{\frac{2}{3}} = 1$$

which is the equation of the evolute to an ellipse.

The caustic formed by rays refracted at a spherical surface does not take a form so simple as these which have been discussed; it can, however, be always described graphically by computing the place of the primary focus.

The properties of diacaustics will be found discussed in the "Treatise on Caustics," by G. H. S. Johnson, M.A., Tutor of Queen's College, Oxford; also in Coddington's "Optics;" in Dealtry's "Fluxions," &c.

In Sir John Herschel's Treatise on 'Light,' in the "Encyclopædia Metropolitana," Art. 170, will be found a list of authors who have written on caustics.

The form of the caustic produced by both refraction and reflexion in a drop of rain, in the case of the rainbow, will be found in the Author's Paper on that subject in PART I. Vol. VI. of the Transactions of the Cambridge Philosophical Society, and it must be also now discussed in the Treatises on Physical Optics.

THE END.

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